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To show that $M^{\perp}$ is closed, we show that if (y_n) is convergent sequence in $M^{\perp}$ then the limit y also belong to $M^{\perp}$ let x ∈ M then the inner-product is continuous and there for

$$\langle x, z \rangle = \langle x, lim_{n\to\infty} y_n \rangle \lim_{n\to\infty} y_n = 0$$

Since $\langle x, y_n \rangle = 0$ for every x ∈ M and $y_n \in M^{\perp}$

This implies that orthogonal complement of a subspace is a closed

t) If H be Hilbert space and satisfy the following condition :

i) For each x ∈ H there is a unique closed , y ∈ M such that $\|x - y\| = \min_{z\in M}\|x - y\|$

ii) The point y ∈ M closed set to x ∈ H is the unique element of M with the property that (x-y) ⊥ M .

Then M is closed linear subspace of Hilbert space H . this is expresses of the fundamental geometrical properties of Hilbert space

k) Let $f_1,\cdots,f_n$∈ H, then vect($f_1 \cdots f_n$) is the closed vector space and subspace of H, generator by linear combinations of $f_1 \cdots f_n$.

If M is a subset of H , then $M^{\perp}$ = { f ∈ H | $\langle f, g \rangle$ } =0, ∀g∈ H is a subspace of H.

If f, g ∈ M and α ∈ C one has (f + αg) ∈ M and if M is closed [any Cauchy sequences in M converges strongly in M], then M is called a a subspace of H this implies that M is linear manifold of a Hilbert space H

j) If x, y are isomorphic topological spaces s.t are contains F: X→ Y and let f: X → f(x) is isomorphic topology on subspace f(x) of onto Y when then we call f is embeds X onto Y, let H is real sequence set (X_n) s.t $\sum_{i=1}^{\infty} X_n^2$ is convergent sequence and let d is metric in H s. d $((X_n), Y_n)) = \sqrt{\sum (X_n - Y_n)^{\ 2}}$

We call (H, d) is Hilbert space and also R^n is isomorphic topology is the Hilbert space.

Conclusion: use geometrical structure of Hilbert space in description quantization process for the quantum state for the particles.

Z ∈ P_cx

if and only if z ∈ C and ∀y ∈ C Re (x-z, y-z) ≤ 0

Proof : Suppose first that z = P_c x. By definition z ∈ C . Let y ∈ C . Since C is convex then ty+(1−t)z ∋ C for all t ∈ [0,1], and since z is the unique distance minimize from x in C :

0> $\|x-z\|^2 - \|x-(ty+(1-t)z\|^2$ =$\|x-z\|^2 - \|(x-z)-t(y-z)\|^2$. Thus , for all 0 < t ≤ 1 Re(x-z,y-z) < $\frac{1}{2}$t $\|y-z\|^2$

Leting t→ 0 we get that Re(x-z,y-z) ≤ 0

Conversely , suppose that z ∈ C and that for every y ∈ C

Re(x-z,y-z) ≤ 0 . For every y ∈ C,

$\|x-y\|^2 - \|x-z\|^2 = \|(x-z)-(z-y)\|^2 - \|x-z\|^2$

$= \|x-z\|^2$ - 2Re(x-z,y-z) ≥ 0

Which implies that z is the distance minimizer, z = P_cx

a) suppose y ∈ A and let x-y ∈ $A^{\perp}$. i.e ,for all u ∈ A

(x- y, u-y) = 0 ≤ 0

Hence y = P_Ax (by above proposition)

Conversely, suppose that y = P_Ax and let u∈ A. By above proposition

(x-y, u-y) ≤ 0

Now enable replace u by (-u). it follows that for all u ∈ A Re(y-x,u)=0

And can obtain im(y-x,u)=0

Now we can say if y ∈ A and x-y ∈ $A^{\perp}$ when y ∈ p_Ax we obtain that A is closed sub space of Hilbert space (H, (· ,·))

d) The inner-product by ⟨·,·⟩, which is another common notation for inner-products that is often reversal for Hilbert space . The inner-product structure of a Hilbert space in geometric way

Example 10: if H be a Hilbert space and A a subset of H. Let x, y, z ∈ $M^{\perp}$ and α , β ∈ C, then there exist linearly of the inner-product ⟹

⟨ x, αy +βz⟩ = α ⟨ x, y⟩ + β⟨x, z ⟩ = 0 for all x ∈ M

There for ⟨ α y, βz⟩ ∈ $M^{\perp}$, so $M^{\perp}$ is linear subspace .

Example 10 : the closed ball B($X_\circ$,r) = { x∈X : $\|x - x_\circ\| \le$r is a closed subset in any normed linear, and hence inner-product , space X.Then ,
$|\|x_n - x_\circ\| - \|x - x_\circ\|| \le \|(x_n - x_\circ) - (x - x_\circ\| = \|x_n - x\| \to 0$
So $\|x_n - x_\circ\| \to \|x - x_\circ\|$ and since $\|x_n - x_\circ\| \le$ for all n ∈ N it follow that $\|x - x_\circ\| \le 1$, so x∈ B[$x_\circ$,r].

Proposition (4): A Hilbert space H is separable iff H has a countable orthonormal basis β ⊂ H. Moreover, if H is separable, all orthonormal bases of H are countable.

Proof. Let D ⊂ H be a countable dense set D = $\{ u_n\}_{n=1}^{\infty}$. By Gram-Schmidt process there exists β = $\{ V_n \}_{n=1}^{\infty}$ an orthonormal set such that span V_n: n = 1, 2 . . . , N} ⊇ span$\{U_n$: n = 1, 2 . . . , N}. So if $\langle x, v_n\rangle$= 0 for all n then $\langle x, u_n\rangle$ = 0 for all n. Since D ⊂ H is dense we may choose $\{w_k\}$ ⊂ D such that x = $lim_{k\to\infty} w_k$ and therefore
$\langle x, x\rangle = lim_{k\to\infty}\langle x, w_k\rangle$ = 0. That is to say x = 0 and β is complete.

Proposition (5): Let (X, ||·||) be a normed space and C ⊂X a convex subset. Then,
i) The closure $\bar{C}$ is convex.
ii) The interior C° is convex.

Comment 2: Interior and closure are topological concepts, whereas convexity is a vector space concept. The connection between the two stems from the fact that a normed space has both a topology and a vector space structure.

Proof:

1) let y ∈ $\bar{C}$.For every $\mathcal{E} > 0$. There are points $x_\varepsilon, y_\varepsilon$ ∈ C with $\|y - y_\varepsilon\| < \varepsilon$ and $\|y - y_\varepsilon\| < \varepsilon$

Let 0≤ t ≤ 1. Then, tx_ε+(1-t)y_ε ∈ C and
$\|(tx + (1-t)y_\varepsilon\| \le$ t$\|x - x_\varepsilon\|$ + (1-t)$\|y - y_\varepsilon\|$ <0
Which implies that tx+(1-t)y ∈ $\bar{C}$, hence $\bar{C}$ is convex

2) x, y ∈ C°. By definition of the interior there exists an r > 0such that
B(x,r) ⊂ C and B(y,r) ⊂ C
Since C is convex ,
∀ ∈ [0,1] tB(x, r)+ (1-t)B(y, r) ⊂ C. But
B (tx + (1-t)y, r) ⊂ tB(x, r) +(1-t) B(y, r),
Which proves that (tx + (1-t)| y ∈ C° is convex. **Comment 3:** Let W ⊂ R^N be a domain and consider the Hilbert space L^2(Ω. The subset of functions that are non-negative (up to a set of measure zero) is convex (but it is not a linear subspace).
are disjoint sets from τ_ω which contain x and z respectively. This shows that (H, τ_ω) is a Hausdorff space. In particular, this shows that weak limits are unique if they exist.

Proposition(4) : Let C be a closed convex set in a Hilbert space (H ,(·, ·)). Then for every x ∈ H ,

$H = H_1 \cdots H_n$

Along with coordinate-wise vector space operations. Considers a product:

$(\cdot,\cdot)_H$ H×H→C:

$((x_1 \ldots x_n), (y_1,\ldots, y_n))_H = \sum_{k=1}^{n}(x_k, y_k)_{H_k}$ and $(\cdot,\cdot)_H$ is an inner-product on H and still convergence in H is equivalent to component-wise convergence in each of the H_k and so $(\cdot,\cdot)_H$ satisfies axioms of is an inner-product on H.

When we use definition of convergent enable H is complete if and only if all the H_k are complete.

b) Suppose H is an infinite dimensional Hilbert space and $\{X_n\}_{n=1}^{\infty}=1$ is an orthonormal subset of H. Then $\| X_m \; X_n \|^2 = 2$

For all m $\neq$ n and in particular, $\{ X_n \}_{n=1}^{\infty}=1$ has no convergent subsequences. From this we conclude that C := {x ∈ H : $\|X\| \leq 1$} , the closed unit ball in H, is not compact. To overcome this problem it is sometimes useful to introduce a weaker topology on X having the property that C is compact.

c) Let (X, $\|x\|$) be a Banach space and X^* be its continuous dual.

The weak topology, τ_ω, on X is the topology generated by X^*. If N $\{ X_n \}_{n=1}^{\infty} \subset X$ is a sequence we will write $X_n \overset{\omega}{\rightarrow} X$ as n → ∞ to mean that $X_n \rightarrow$ x in the weak topology. Because $\tau_\omega = \tau(X^*) \subset \tau_{\|\cdot\|} := \tau(\{\|x - \cdot\|$:x ∈ X} , it is harder for a function f : X → F to be continuous in the τ_ω- topology than in the norm topology $\tau_{\|\cdot\|}$

In particular if φ : X → F is a linear functional which is τ_ω continuous, then φ is $\tau_{\|\cdot\|}$ continuous and hence φ ∈ X^*.

Proof. By definition of τ_ω, we have $X_n \overset{\omega}{\rightarrow}$ x ∈X iff for all $\beta \subset\subset X^*$ and ∈ > 0 there exists an N ∈ N such that |φ(x) − φ(X_n)| < ∈ for all n ≥ N and φ ∈ Γ.

This later condition is easily seen to be equivalent to:

φ(x) = $\lim_{n\rightarrow\infty} \varphi(x_n)$ for all φ ∈ X^* .

The topological space (X, τ_ω)is still Hausdorff, however to prove this one needs to make use of the Hahn Banach Theorem . For the moment we will concentrate on the special case where X = H is a Hilbert space in which case

$H^* = \{\varphi_y := \langle 0 , y\rangle$: y ∈ H} , if x, y ∈ H and y := z −x $\neq$ 0, Then

0< ∈:= { ω ∈ H : $|\varphi_y(x) - \varphi_y(\omega)| < \frac{\in}{2}$ } and

V_y := {ω ∈ H :$|\varphi_y(z) - \varphi_y(\omega)| < \in$

2. Any translate of a dilate of a convex set is itself a convex set; that is, if C is convex then so too is: x+ αx := {x+αc: c∈ C }for all x∈ X and α∈C
3. From 2 we see that any affine set; that is a translate of a subspace, is convex. Taking R3 as an example, in which the only subspaces are the set containing just the origin, lines and planes passing through the origin, this implies that any one point set, line or plane in R3 is a convex subset.
4. The unit ball, and hence any closed ball, in a normed linear space is a convex subset, as is any open ball. Verification of each of the above is left as an exercise.

Convergent sequences A sequence of points of X, $x_1, x_2, \cdots x_n$ will be denoted by $\{ X_n \}_{n=1}^{\infty}$=1, or simply (x_n) when the context makes it clear that we are talking about a sequence. Formally, we regard the sequence (xn) as a function x:N → X : N → X(n) = x_n

Theorem: Suppose that H is a Hilbert space and M ⊂ H be a closed convex subset of H. Then for any x ∈ H there exists a unique y ∈ M such that $\|x-y\|$= d(x, M) = $inf \ \|x-z\|_{\ z\in M}$[1]

Definition (11): Let H is a Hilbert space and M ⊂ H be a closed subspace.

The orthogonal projection of H onto M is the function: P_m: H → H such that for x ∈ H, P_m (x) is the unique element in M such that (x − P_m(x)) ⊥ M.

Definition(10) : (Basis). Let H be a Hilbert space. A basis β of H is a maximal orthonormal subset β ⊂ H.

Proposition (3). Every Hilbert space has an orthonormal basis.

Lemma2: Let β be an orthonormal subset of H then the following are equivalent:
(1) β is a basis,
(2) β is complete and
(3) $\overline{span\ \beta}$ = H[1].

Proposition(4) :
Any an atas U= $\{(U_{\alpha,}\emptyset_\alpha)\}$ on alocally Euclidean space is contained in unique maximal atlas.

In summary, to show that a topological space M is a C^∞ manifold, it suffices to check that
1) M is Housdorff and second countable
2) M has a C^∞ atlas

5-Geometrical structure of Hilbert space

a) Let $H_1\ H_2\ \cdots\ H_n$ be a finite collection of inner-product spaces. We can Define the space

1[DR] J. Derezi´nski, S. Richard, *On almost homogeneous Schrödinger operators*, Preprint arXiv:1604.03340

$(x,y) \rightarrow \sum_i^n x_i y_i$

is an inner product. The induced metric

$d(x,y) = (\sum_i^n X_i - Y_i)^{\frac{1}{2}}$

Is called the Euclidean metric. It is known that R_n is complete with respect to this metric, hence it is a Hilbert space (in fact, any finite-dimensional normed space is complete, so that the notion of completeness is only of interest in infinite-dimensional spaces).

4- Weak Convergence. Suppose H is an infinite dimensional Hilbert space and $\{ X_n\}_{n=1}^{\infty}$=1 is an orthonormal subset of H. Then, $\{ X_n \quad X_m \}^2 = 2$ for all m ≠ n and in particular, $\{ X_n \}_{n=1}^{\infty}$ has no convergent subsequences. From this we conclude that C: = {x ∈ H: $\|X\| \leq$ 1} , the closed unit ball in H, is not compact. To overcome these problems it is sometimes useful to introduce a weaker topology on X having the property that C is compact.

Example (10): To see an example of a ball in a space where the norm is not induced by an inner-product, disk, it is diamond-shaped:

5- Convex sets

We now wish to describe when a subset of a linear space is convex (intuitively, when its boundary is always `bowed' outward). To do this, we first need to capture a precise definition of the line segment between two elements of a linear space.

Definition (9): Let x and y be elements of a linear space X. The line segment joining x and y is:

[x, y] :={(1- $\alpha)x$ + α y}: α ∈[0,1]

Remark 3: As the variable increases from 0 to 1, the vector {(1- α) + }y traces out the points between x and y lying on the straight line through them.

Definition: A subset C of a linear space X is convex if whenever x; y ∈ C then [x, y] ⊂ C. That is, if two points lie in the set then necessarily so does the line segment joining them. **Proposition** (2) — Convexity is closed under intersections. Let V be a vector space. Let $\{U_a$⊂ V | a ∈ A} be a collection of convex sets (not necessarily countable). Then U = $\cap_{a \in A} U_a$ is convex.

Example12:

1. Any subspace of convex is convex.

3- HILBET SPACE

Definition (6): Hilbert space. A complete inner-product space is called a Hilbert space. (Recall: a space is complete if every Cauchy sequence converges.)

Comment (2): An inner-product space (H, (·, ·)) is a Hilbert space if it is

$$D(x, y) = (x - y, x - y)^{\frac{1}{2}}$$

Completeness is a property of metric spaces. A sequence (X_n) ⊂ H is a Cauchy sequence if for all e > 0 there exists an N ∈ N such that for every m, n > N:

$$\|X_n - X_m\| < \in$$ [1]

Theorem (3) : Completion. Let (G, $(\cdot,\cdot)_G$) be an inner-product space. Then, there exists a Hilbert space (H, $(\cdot,\cdot)_H$), such that:

1) There exists a linear injection T : G → H, that preserves the inner-product, $(X, Y)_G = (Tx, Ty)_G$ for all x,y ∈ G (i.e., elements in G can be identified with elements in H).

2) Image (T) is dense in H (i.e., G is identified with "almost all of" H).

Moreover, the inclusion of G in H is unique: For any linear inner-product preserving injection T_1 : G →H_1 where H_1 is a Hilbert space and Image(T_1) is dense in H1, there is a linear isomorphism S :H →H_1, such that T_1= S ○T (i.e., H and H_1 are isomorphic in the category of inner-product spaces). In other words, the completion G is unique modulo isomorphism.

Definition (9): Let H is a Hilbert space and M ⊂ H be a closed subspace. The orthogonal projection of H onto M is the function P_M : H → H such that for x ∈ H, P_M (x) is the unique element in M such that

$(x - P_M(x)) \perp M$.[1]

Example (9) of Hilbert spaces

The space R^n is a real vector space. The mapping

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[1] [DR] J. Derezi´nski, S. Richard, *On almost homogeneous Schrödinger operators*, Preprint arXiv:1604.03340

B[x] = {(x,y) ∈ $\mathbb{R}^2$: | x| + | y≤ 1 }[1]

So the geometry of the ball tells us about the structure of the normed space.

Theorem 1: (Schwarz Inequality). Let (H, (·| ·)) be an inner product space, then for all x, y ∈ H

| (x, y) | ≤ | | x| | | | y| |

and equality holds iff x and y are linearly dependent.

Corollary (1) Let (H, ⟨· | ·⟩) be an inner product space and $\|x\|:=\sqrt{\langle x, x\rangle}$

Then

|| · || is a norm on H. Moreover ⟨·, ·⟩ is continuous on H ×H, where H is viewed as the normed space (H, || · ||).

verifier : The only non-trivial thing to verify that || · || is a norm is the triangle inequality:

$\| x + y \|^2 = \| x \|^2 + \| y \|^2 + 2\mathrm{Re}\langle x, y\rangle \leq \| x\|^2 \| y \|^2 + 2 \| x\| \| y\|$

$= (\| x\| + \| y\|)^2$

Theorem (2) . An inner product space V is a metric space with the distance function given by

D(u, v) = $\|\boldsymbol{u} - \boldsymbol{v}\|$

Proof: let w= u-v :then form the requirement that

$\|u - v\| \geq \mathbf{0}$ and || u-v|| =0 ↔ u =v

Similarly, from the requirement that|| w|| = || -w|| , it follows that

|| u-v|| = || v-u||

Finally, let **u** be replaced by **u-w** and **v** by **v−w** in the triangle inequality

|| u-w|| ≤ || u-v|| +|| u-w ||

Which is the third and last requirement for a distance function in a metric space.[1]

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[1][Yaf] D.R. Yafaev, *Mathematical scattering theory. General theory*, Translations of Mathematical Monographs **105**, American Mathematical Society, Providence, RI, 1992

That is, the circular disk, ($x^2 - x_\circ^2 + y^2 - y_\circ^2$) $\leq r^2$. If a weighted inner- product is used, elliptical disks are obtained instead of circular ones

Example 7 : In $\mathbb{R}^3$ with (x | y) := x.y the closed ball centre ($x_\circ$, $y_\circ$) radius r is a solid sphere , or *ball* . With a weighted inner-product it will be an ellipsoidal ball. Of special interest is the unit ball of X, B[X] = B[0; 1], the closed ball centered at the origin of radius one. Knowledge of the unit ball provides knowledge of every other ball in the linear space via the following relation:

B[$x_\circ$; r] = $x_\circ$+ rB[X] := { x ∈ X : x = $x_\circ$+ ry for some y ∈ B[X]}

One can also determine the norm of a space (and then, through the polarization identity, determine the inner-product if the norm satisfies the parallelogram law) from a knowledge of the unit ball via the following expression:

$$\|x\| = \inf\{\alpha\ 0\ : x \in \alpha\, \beta[X]\ (\text{or equivalent } \tfrac{1}{\alpha} x \in \beta[X])\}$$

Definition (5): Metric space. A metric space is a set X , endowed with a function d :X ×X →R, such that Positivity: d(x,y) ≥ 0 with equality iff x = y. ! Symmetry: d(x,y) = d(y,x). Triangle inequality:

d(x,y) ≤ d(x, z)+d(z,y).

Please note that a metric space does not need to be a vector space. On the other hand, a metric defines a topology on X generated by open balls,

B(x, r) = {y ∈X ‖ d(x,y) < r}.

As topological spaces, metric spaces are Para compact (every open cover has an open refinement that is locally finite), Hausdorff spaces, and hence normal (given any disjoint closed sets E and F, there are open neighborhoods U of E and V of F that are also disjoint). Metric spaces are first countable (each point has a countable neighborhood base) since one can use balls with rational radius as a neighborhood

Example 8: to see an example of a space where the norm is not induced by an inner –product, consider X= $\mathbb{R}^2$ with the norm | | (x, y| | := | x| + | y| . Here , instead of the unit ball being a circular disk, it is diamond-shaped:

$(\alpha f)(x) := \alpha f(x) \quad \forall f \in C([a;b];C) \quad \forall \alpha \in C \quad \forall x \in [a;b]$

The standard inner-product is defined as:

$$< f \mid g> \;=\; \int_a^b f(x)\overline{g(x)}\,dx$$

Note, a weighted inner-product on this space is also possible:

$$< f \mid g> \;=\; \int_a^b w(x)\, f(x)\overline{g(x)}\,dx$$

Where w is real valued function on [a; b][1] which continuous and strictly positive except at possibly a finite number of points.

2- Closed and Open Balls

Definition(4): Let X be an inner-product space

(with norm $\|x\| := \sqrt{(x\, x)} \quad \forall x \in X$)

1. The closed ball with centre x0 and radius r > 0 is:

$B[x_\circ, r] := \{ x \in X : \|x - x_\circ\| \le r$

2. The open ball with centre $x_\circ$ and radius r > 0 is:

$B(x_\circ, r) := \{x \in X : \|x - x_\circ\| \le r$

3. The sphere with center $x_\circ$ and radius $r > 0$ is

$S(x_\circ, r) := \{x \in X : \|x - x_\circ\| = r$[1]

Example 5 : consider $\mathbb{R}$ with $(x \mid y) = xy$ and $\|x\| = \sqrt{x^2} = |x|$ the absolute value of x . The open ball center $x_\circ$

$B(x_\circ, r) = \{ x \in \mathbb{R} : | x - x_\circ| < r \}$

That is, the open interval $(x_\circ - r, x_\circ + r)$

Example 6: consider $\mathbb{R}^2$ with $(x \mid y) := x \cdot y$ and $\| x \| = \sqrt{x^2 + y^2}$.

The closed ball center $(x_\circ, y_\circ)$ r radius is then

$B(x_\circ, r) = \{ (x, y) \in \mathbb{R}^2 : \sqrt{(x^2 - x_\circ^2) + (y^2 - y_\circ^2)} \le r$

[1] [Yaf] D.R. Yafaev, *Mathematical scattering theory. General theory*, Translations of Mathematical Monographs **105**, American Mathematical Society, Providence, RI, 1992[1]

N2) ‖ α x ‖ = | α | ‖x ‖ ∀ x ∈ X, α ∈ C [1]

N3) ‖ x + y ‖ ≤ ‖ x ‖ + ‖ y ‖ ∀ x, y ∈ X[2]

Remark(2): A linear space X equipped with a norm is sometimes referred to as a normed linear space.

Intuitively ‖x ‖ represents the `length' of the vector x. The term `scalar' may derive from the way in which positive scalar quantities scale the lengths of vectors up or down, as per N2.

The property N3) is often referred to as the Triangle Inequality or 'Minkowski's inequality'.

Example 2: C^n with the usual dot-product

(x | y) := $X_1\overline{Y_1}$ + ….. +$X_n\overline{Y_n}$ = x ·y.

Definition(3): Let X be an inner-product space. The norm induced by the inner-product (x | y) is the Function ‖· ‖ : X → R defined by:

‖x ‖ := $\sqrt{(x\, x\,)}$ ∀ x ∈ X

Further examples of inner-product spaces

So far we have two examples of an inner-product space:

Example 3: $C^n$with the usual dot-product

(x | y) := x_1 $\overline{y_1}$+ ···+ $x_n\bar{y}_n$= x ·y. Some further examples are:

Example 4: C(| a, b|) or C(| a, b |) , R the space of all continuous complex-valued (real valued)functions defined on a real interval [a; b] (where a < b). When the scalar field over which we are working is clear we will sometimes simply write C[a; b] instead of C([a; b];C) (or C([a; b];R). The vector operations on these spaces are defined point wise in the following manner:

(f + g)(x) := f(x) + g(x) ∀ f; g ∈ C([a; b];C) ∀x ∈[a; b]

1] BM] H. Baumg¨artel, M. Wollenberg, *Mathematical scattering theory*, [1] Birkh¨auser verlag, Basel, 1983.

1-Basic concepts

Definition(1): Let X be a linear space. An inner-product on X is a function (. , .) : H X H → C (recall) which satisfies the following:

1) (x, x) ≥ 0 and (x , x) = 0 ↔ x = 0 ∀x ∈ X

2) (y, x) = $\overline{(x , y)}$ ∀ x; y ∈ X

3) (x + y, z) = (x, z) + (y, z) ∀ x; y; z ∈ X

4) (αx ,y) = α (x , y) ∀x; y ∈ X; ∀α ∈ C

Comment (1): By definition, every linear subspace is closed under vector space operations (it is algebraically closed). This should not be confused with the topological notion of closeness', which is defined once we endow the vector space with a topology. A linear subspace may not be closed in the topological sense.

Remark (1): Alternative notations for the inner-product include (x; y) and the 'bra-ket' notation, which is frequently adopted in Physics, namely < x‖y > or < x | y >.

IP3 states that an inner-product is additive in the first variable, while IP4 states that it is scalar Homogeneous in the first variable, taken together they assert that it is a linear function of its first[1]

Variable. **Lemma** (1) Let X be a linear space equipped with an inner-product < · | ·>. Then:

1) < x | y + z> = < x | y> + < x | z> = ∀x ,y, z ∈ X

2) < x | αy> = $\bar{\alpha}$ < x | y>. ∀ x ,y ∈ X

3) < · | y> = 0 ∀ x ∈ X[1]

Definition (2): Let X be a linear space. A norm on X is a function ‖x ‖: X → R which satisfies the following:

N_1) ‖x ‖ ≥ 0 and ‖x ‖= 0 ↔x = 0, ∀ x ∈ X

ملخــــص

لفضاءات هلبرت أهمية كبيرة في الرياضيات والفيزياء الكمية. وأيضاً يعتبر تعميم طبيعي للفضاءات المتجهية المنتهية. هدف الدراسة هو إيجاد هيكل هندسي رياضي من فضاء هلبرت. توصلت الدراسة إلى أنه يمكن بناء هياكل هندسية رياضية من فضاءات هلبرت باستخدام فضاءات المتسلسلات ومفهوم متعدد الطيات التبولوجى. توصي الدراسة بأنه يمكن استخدام هذا البناء الهندسي في وصف المعالجة للحالة الكمية للجسيمات.

Abstract

The Hilbert spaces were very important mathematical and quantum physics. Also were purposes of the most natural generalization of finite dimensional geometry to vector space, which need not have a finite basis. Our study aim to find geometrical structure of a Hilbert space using the mathematical concepts. Results of showed that enable find mathematic geometrical structure of a Hilbert space by using sequential spaces and topological manifold concept. We concluded that study recommended use geometrical structure of Hilbert space in description quantization process for the quantum state for the particles.

The Geometrical structure of Hilbert space

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