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Now: if we have a basis Lie g a algebra:

$So(3) = (x_1, x_2, x_3)_k$or some basis x_i one simply allows linear combinations over **C** i.e. $g \cong R^n$by $g \cong C^n$ extending the commutation relation $So(3)_c = (x_1, x_2, x_3)_c$ from now on we work with lie algebras over **C** which is very useful and much easier than R, then finite rotations are given by$e^u = e^{lu_k/k} = (Rc\vec{u}) \in so(3)$

Similarly, consider so(2). According to the above, its lie algebra is

$So(2) = \{A \in Mat(u, c); A^T = -A = 0, Tr(A) = 0\}$

We denote to a basis of su(2) by the pauli matrices,$X_i = \frac{1}{2}\sigma_i$ for

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Hence any$u \in su(2)$can be written uniquely as$u = u_j(i\sigma_j)$. Then

$$e^u = e^{uj(i\sigma_j)} \epsilon su(2)$$

Again, one defines the complex field generators$J_k = \frac{1}{2}\sigma_i$

Which satisfy$[J_i, J_j] = i \in_{ijk} J_k [x_i, x_j] = -\in_{ijk} X_k$ then for$so(3) \cong su(2)$

Tensor product

If V and W are 2 representations of the lie group G, then so are$V \otimes W$by

$$\mathrm{T} = \mathrm{G} \rightarrow \mathrm{GL}(V \otimes W)$$

$$g \rightarrow T_v(g) \otimes T_w(g)$$

Passing to the lie algebra by differentiating, this becomes g= $e^{ti[\sigma^N j_N]}$,

$$\mathbf{T} = g \rightarrow gl(v \otimes w) = gl(v) \otimes gl(w)$$

$$g \rightarrow T_v(g) \otimes |+| \otimes T_w(g)$$

It is easy to check directly that this is a representation of g.

This meaning of adding angular momenta in quantum mechanics via

$$J_i = L_i + S_i$$

For semi-simple lie algebras.

Thus we reproducing finite rotation matrices

$$\overrightarrow{R(\emptyset e_x)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\emptyset & sin\emptyset \\ 0 & -sin\emptyset & cos\emptyset \end{pmatrix} = \overrightarrow{e^{i\emptyset e_x J}} = e^{i\emptyset J_x}$$

Let see if this is true: using

$$J_x^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

we get

$$e^{i\emptyset J_x} = \sum_{m=1}^{\infty} \frac{(i\emptyset J_x)^m}{m!} = 1 + J_x^2 \sum_{n=1}^{\infty} \frac{(i\emptyset)^{2n}}{(2n)!} + J_x \sum_{n=0}^{\infty} \frac{(i\emptyset)^{2n+1}}{(2n+1)!}$$

$$= 1 + J_x^2(cos\emptyset) - 1 + iJ_x sin(\emptyset) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\emptyset & sin\emptyset \\ 0 & -sin\emptyset & cos\emptyset \end{pmatrix}$$

Example: consider so(3) we define

$So(3) = \{A \in Mat(u, R); A^T = -A, Tr(A) = 0\}$

A convenient basis of $So(3)$is given by

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence any$u \in so(3)$ can be written uniquely as $u = u_K x_k$, and any element of so(3) can be written as$e^u = e^{u_K x_k} \in so(3)$

Their Lie algebra is$[X_i, X_j] = -\epsilon_{ijk} X_k$

It is easy to calculate the exponentials explicitly, reproducing finite rotation matrices in physics; one coefficient allows complex coefficients, defining

Which $J_k = -iX_k$

Which are Hermitian$J_i^t = J_i$ and satisfy the rotation algebra?

$$[J_i, J_j] = i \in_{ijk} J_k$$

$$[J_i^t, J_j^t] = i \in_{ijk} J_k^t$$

Let us calculate the corresponding left-invariant vector field:

$X_g^A = dL_g(X_e^A)$ (5.8) .

We can use the same coordinates near e and g, so that the map L_g has the "coordinate expression

$$L_g^{ij} = (gx)^{ij} = (g)^{ik}(x)^{kj} \quad (5.9)$$

We have

$$\frac{\partial}{dL_g \partial x^{ij}|_e} = \frac{\partial (L_g x)^{kl}}{\partial x^{ij}}\bigg|_e \frac{\partial}{\partial x^{kl}|_g} = \frac{\partial g^{km}x^{ml}}{\partial x^{ij}|_e} = \delta^{ij} g^{ij} \frac{\partial}{\partial x^{kl}|_g} = g^{kj} \frac{\partial}{\partial x^{kl}|_g} \text{ (5.10)}$$

Therefore for general $X_e = A_{ij} \frac{\partial}{\partial x^{ij}|_{e}}$ we have

$$X_g^A = dL_g(X_e^A) = g^{ki} A_{ij} \frac{\partial}{\partial g^{kl}} \quad (5.11)$$

Where we write $g^{ij} = x^{ij}(g)$ for the commutator. Let X^A, X^B be left- invariant vector fields as above. Noting that

$$\frac{\partial}{\partial g^{kl}}\bigg|_g g^{ki} = \delta^{ik}\delta^{jl} \quad (5.12)$$

and using (5.11) we have

$$[X^A, X^B] = g^{ki} A_{ij} \frac{\partial}{\partial g^{kj}} g^{ki} B_{ki} \frac{\partial}{\partial g^{ki}} g^{ki} B_{ij} \frac{\partial}{\partial g^{kj}} g^{ki} A_{ki} \frac{\partial}{\partial g^{ki}}$$

$$= g^{ki} A_{ij} B_{Jj} \frac{\partial}{\partial g^{jj}} g^{ki} B_{ij} A_{jj} \frac{\partial}{\partial g^{ki}}$$

$$= g^{ki}(A_{ij}B_{Jj} - B_{ij}A_{Jj}) \frac{\partial}{\partial g^{kl}} = [A, B]_{ij} \frac{\partial}{\partial g^{kj}} = X^{[A,B]} \text{ (5.13)}$$

We used some concepts of differential geometry to find geometric representations of Lie Group and Lie Algebra. Finally the equations (5.8), (5.10), (5.11) (5.12) and (5.13) be geometrical representation of Lie Group and Lie Algebra.

Now we consider that

$$J_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Which is a rotation generator i.e $J_x \in so(3)$, we claimed previously that

$$sp(n) = \{A \in gl(n,R), A^T = |A|\}$$

where the Lie algebra is again defined by the commutator.

Also, the space of vector fields on a manifold M together with the Lie bracket forms an infinitesimal Lie algebra.

Structure constants: Let g be a finite-dimensional Lie algebra, and let X_1, X_n be a basis for g (as a vector space). Then for each i, j, $[X_i, X_j]$ can be written uniquely in the form

$$[X_i, X_j] = \sum_{k=1}^{n} C_{ij}^k X_k \quad (5.4)$$

The constants C_{ij}^k are called the structure constants of g (with respect to the chosen basis). Clearly, the structure constants determine the bracket operation on g. (Often in physics one uses ig in order to have hermitian generators, which leads to

$$[X_i, X_j] = i \sum_{k=1}^{n} C_{ij}^k X_k \quad (5.5)$$

The structure constants satisfy the following two conditions,

$C_{ij}^k + C_{ji}^k = 0$ (antisymmetry)

$$\sum_{k=1}^{n} \left(C_{mk}^{m} + C_{jk}^{m} C_{mi}^{l} + C_{ki}^{m} C_{mj}^{l}\right) X_k \quad (5.6) \text{ (Jacobi identity)}$$

The Lie algebra of $GL(n,R)$:

Recall that $GL(n,R)$: is an open subset of R^{n^2}. A natural coordinate system on GL(n, R) near the unit element $e = 1$ is given by the" Cartesian matrix coordinates",

$$X^{ij}(g) = g^{ij} \quad (\text{i. e } x: GL(n) \to R^{n^2}$$

Where $g = g^{ij}$ .now we ca say that basis of tangent vectors $T_e(GL(n))$ is then given by the partial derivatives $\left.\frac{\partial}{\partial x^{ij}}\right|_e$, i.e a general tangent vector at e has the form

$$X_e^A = \left.\frac{\partial}{\partial x^{ij}}\right|_e \quad A_{ij} \in R \ (5.7)$$

(sum convention). Hence $T_e(GL(n)) = Mat(n,R) = gl(n,R)$ as the vector space. Denote with $gl(n) = Mat(n)$ the space of $n \times n$, matrices. We want to show that

$$Lie(GL(n)) = gl(n)$$

As Lie algebras (with the commutator for $gl(n)$), not just as vector spaces; the latter is evident.

A Homomorphism between Lie groups is a smooth map $\emptyset: G \rightarrow H$ which is a group homomorphism.
If $H = GL(V)$ for some vector space V, this is called a representation of G. One considers in particular the following types of representations:

$\pi: G \rightarrow GL(n, R)$ n - dimensional"real" representation

$\pi: G \rightarrow GL(n, R)$ n - dimensional"complex" representation

$\pi: G \rightarrow U(n)$ n - dimensional"unitary" representation

A Lie algebra g over R (resp. C etc... any field) is a vector space over R respect and an operation (a Lie bracket

$$[\ ,\ ]: g \times g \rightarrow g$$

Which is bilinear over R (resp. C) and satisfies

$$[X, Y] = 0$$

and the Jacobi identity

$$[X, [Y, Z]] + [Z, [X, Y]] + [Z, [Y, X]] = 0 \quad (5.3)$$

The first property implies

$[X, Y] = -[Y, X]$ = "antisymmetry"

Note: for any associative algebra A, there is an associated Lie algebra g, which is A as a vector space and

$[X, Y] = XY - YX$"commutator" The Jacobi identity is then trivial

Examples 5.1: let

$$gl(n, R) = Mat(n, R) = Mat(n \times n, R)$$

with $[X, Y] = XY - YX$

The following Lie algebras are particularly important

$$sl(n, R) = \{A \in gl(n, R), Tr(A) = 0\}$$

$$so(n) = \{A \in gl(n, R), A^T = -A, Tr(A) = 0\}$$

$$u(n) = \{A \in gl(n, R), \quad A^T = -A\}$$

$$su(n) = \{A \in gl(n, R), \quad A^T = -A, \quad Tr(A) = 0\}$$

definition4.3: Lie brackets of vector fields: Let $X;Y$ vector fields on . Then one cans define a new vector field

$[X,Y]$ on M via

$[X,Y]p(F) = X_p[Y[f]] - Y_p[X[f]]$ (4.3)

One then easily shows

Theorem 4.2 :

$[X,Y]$is indeed a vector field on M (derivation!)

- $[X,Y] = 0$, hence$[X,Y] = -[X,Y]$
- $X,[Y,Z] + [Z,[X,Y]] + [Z,[Y,X]] = 0$

$$[fX,gY] = fg\,[X,Y] + fX[g]Y - gY[f]X$$

Given a map $\emptyset: M \to N$, let$\tilde{X},\tilde{Y}$, $\emptyset$-related to$[\tilde{X},\tilde{Y}]$, be vector field on such that $d\emptyset(X) = \tilde{X}$ and $d\emptyset(Y) = \tilde{Y}$

$$\frac{\partial Y^j(x)}{x^i(x)\partial x^i}\frac{\partial}{\partial x^j} - \frac{\partial x^i(x)}{Y^j(x)\partial x^j}\frac{\partial}{\partial x^i} = 0 \quad (4.4)$$

Because the partial derivatives commute.

5-lie group

A Lie group G is a group which is also a differentiable manifold, such that the maps

$$\mu: G \times G \to G$$

$$(g_1 g_2) \to g_1 g_2 \quad (5.1)$$

And

($V: G \to G$ related", etc) Then $[X,Y]$ is $\emptyset$-related to$[\tilde{X},\tilde{Y}]$,

$$\text{ie } d\emptyset([X,Y]) = d\emptyset(X).d\emptyset(Y).(12)$$

In particular, the space of all vector Fields is a (infinite-dimensional) Lie algebra!

In coordinate system, we can write the vector field as $X = \frac{\partial}{x^i \partial x^j}$ and similar Y. Then

$$[X,Y] = \frac{\partial}{x^i(x)\partial x^i(x)}\frac{\partial}{\partial x^j} - \frac{\partial}{Y^j(x)\partial x^j}\frac{\partial}{x^i(x)\partial x^i} \quad (5.2)$$

$$g \to g^{-1}$$

SU(2) onto SO(3), and which is two-to-one. This is a nice illustration of the importance of global aspects of Lie groups.

4-Differentiable manifolds:

Here is a very short summary of the definitions and main concepts on differentiable manifolds. This is not entirely precise. The proofs can be found e.g. in [Warner]

Definition 4.1: A topological space M is a m-dimensional differentiable manifold if it comes with a family $\{f(U_i, \varphi_i)\}$of coordinate systems ("charts") such that

(1) $U_i \subset M$ open, $U_i U_i =$ $U_i M U_i U_i = U_i M$ and $\varphi_i : U_i \to V_i \subset R^m$s a homeomorphism (=continuous and invertible)

(2) $\varphi_i \varphi_j^{-1}$ smooth (C^∞)where defined$x^m(p)$

Notation:

$$\varphi_{(p)} = \begin{pmatrix} x^1(p) \\ \vdots \\ x^m(p) \end{pmatrix}, \; p \in M \quad (4.1)$$

Definition4.2: smooth maps are

$$C^\infty(M) = \{f : M \to R,\}$$

$$C^\infty(M) = \{f : M \to R, C^\infty\} \quad (4.2)$$

he latter means thatφ_Nf φ_M^{-1} is smooth for all C^∞ coordinate systems if defined.

A smooth invertible map between manifolds$\emptyset : M \to N$ is called a diffeomorphism.

5-Tangential space: let $p \in M_n$. The tangential space of M a p is defined as the space of all derivationsn (="directional derivatives") of functions at p, i.e.

$$T_p M = \{X : C^\infty(M) \to R \; derivation\}$$

Which means that

$$X[\lambda f + \mu g] = [\lambda X[f] \; + \; \mu \times [g], \qquad f, g \in \; C^\infty(M), \qquad \lambda \mu \in R$$

$$X[fg] \; = f(p) \; [X[g] + g(p) X[f] \quad (4.3)$$

In particular, this implies $X[c] \; = \; 0$

all $v_1, v_2 \in C_n$ i.e., that U preserves the inner product on, By taking the determinant of the condition $U^*U = 1$, we see that $|\det U| = 1$ for all $U \in$

$U_{(n)}$. In this, the finite-dimensional case, the condition $U^*U = 1$ implies that U^* is the inverse of U and thus that $U^*U = 1$. This result does not hold in the infinite-dimensional case.

Now we have example (3): A $n \times n$ real matrix $R \in M_n(R)$ is said to be orthogonal if $R^{tr}R = 1$. A matrix R is orthogonal if and only if

w.v= 0

for all $v, w \in R^2$ The group of orthogonal matrices is denoted $O(n)$ and is called the $(n \times n)$ orthogonal group. The special orthogonal group, denoted $SO(n)$, is the subgroup of $O(n)$ consisting of orthogonal matrices with determinant 1.

Comment: the condition $R^{tr}R = 1$ implies that $RR^{tr} = 1$ and that the columns of R from an orthonormal set in R^n Geometrically,

a real matrix R is in $O(n)$ if and only if $R_{v_1}, R_{v_2} = v_1, v_2$ for all $v_1, v_2 \in R^n$ i.e., if and only if R preserves the inner product on R^n. By taking the determinant of the condition $R^{tr}R = 1$ we see that $det\ R = \pm 1$ for all $R \in O(n)$.

Lie groups in physics

3.1 The rotation group $SO(3)$ and its universal covering group $SU(2)$

SO(3) is the rotation group of R^3 which is relevant in classical Mechanics. It acts on the space R^3 as $SO(3)\ R^3$ as

$$SO(3) \times R^3 \to R^3$$

$$(g, x) \to gx \qquad (3.3)$$

In particular, this is the simplest of all representations of SO(3), denoted by

$$\pi_3 : SO(3) \to GL(R^3) \quad (3.4)$$

If a physical system is isolated, one should be able to rotate it, i.e. there should be an action of $SO(3)$ on the space of states M (=conguration space). In Quantum Mechanics, the space of space V (the Hilbert space), which therefore should be a representation of $SO(3)$.

It turns out that sometimes (if we deal with spin), $SO\ (3)$ should be" replaced" by the "spin group" $SU(2)$. In fact, $SU(2)$ and $SO(3)$ are almost (but not quite!) isomorphic. Moreprecisely, there exists a Lie group homomorphism $\phi : SU(2) \to SO(3)$ which maps suggests, most one-parameter unitary groups that arise in applications are not continuous in the operator norm topology.

Before giving the proof of Stone's theorem, let us work out the generator of the group in [$let\ H = L^2R^n$] and let $U_a(t)$ be the translation operator given by

$$(U_a(t)\varphi(x) = \varphi(x + ta))$$

Then $U(\cdot)$ is a strongly continuous one –operator unitary group

Definition 3.2: If G_1 and G_2 are matrix Lie groups, then a Lie group homomorphism of G_1 to G_2 is a continuous group homomorphism of G_1 into G_2 . A Lie group homomorphism is called a Lie group isomorphism if it is one-to-one and onto with continuous inverse. Two matrix Lie groups are called isomorphic if there exists a Lie group isomorphism between them. {Hall, 2015 #7}

Example 3.2: The real general linear group, denoted $GL(n, R)$, is the group of invertible $n \times n$ matrices with real entries. The groups $SL(n, C)$ and $SL(n, R)$ are, respectively, the groups of complex and real matrices with determinant 1. They are called the special linear groups.

Example 3.3: An $n \times n$ matrix $U \in M_n(C)$ is said to be unitary if

$U * U = I$. A matrix U is unitary if and only if

$U_v , U_w = v, w$ for all $v, w \in C^n$. The group of unitary matrices is denoted $U(n)$ and called the $(n \times n)$ unitary group. The special unitary group, denoted $SU(n)$, is the subgroup of $U(n)$ consisting of unitary matrices with determinant 1.

Definition 3.3: Orthogonal vectors. We say that two vectors are orthogonal, or perpendicular, if their inner product is 0. {Hall, 2015 #7}

The condition $(U^*U_{jk} = \delta_{jk})$ is equivalent to the condition that the (12)

Example 3.4:

The vectors $v = \begin{bmatrix} i \\ i \end{bmatrix}$ $u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$: $vu = 0$, this implies that v and u are orthogonal i.e $v , u = 0$, in quantum mechanics if we determine the quantum state u and v to any particle must be are orthogonal.

Columns of U form an orthonormal set in C_n , as can be seen by direct computation. Geometrically, the condition $U^*U = 1$ is equivalent to the condition that $U_{v_1}, U_{v_2} = v_1, v_2$ for

$[X, Y] = XY - YX$ belongs to G.

Example 3.1: The Lie algebras u(n) and su(n) of U(n) and SU(n) are given by:

$$U(n) = \{X \in M_n | X^* = -X\}$$

$Su(n) = \{X \in u(n) \mid trace(X) = 0\}$. The Lie algebra of $O(n)$ is equal to so(n). Finally, the Lie algebra of $O(n)$ is equal to $so(n)$.

Proof: If $X^* = -X$, then by Property 2 of the above Theorem

Theorem

$$(e^{tX})^* = e^{tX^*} = e^{-tX} = (e^{tX})^{-1} (3.1)$$

Showing that e^{tX} is unitary. In the other direction, if $e^{tX}$is unitary for all $t \in R$, then $(e^{tX})^* = (e^{tX})^{-1}$

. Thus,$(e^{tX})^* = (e^{tX})^{-1}$ Differentiating this relation at $t = 0$, using (2.4) gives $X^* = -X$. Thus, the Lie algebra of the first three properties of g say that g is a real vector space. Since$M_n(c)$ is an associative algebra under the operation of matrix multiplication, the last property of g shows that g is a real Lie algebra Prosision (3.1)For a particle moving inR^2 the Hamiltonian flow generated by the angular momentum function

$$J(x, p) = x_1p_2 - x_2p_1$$

Consists of simultaneous rotations of $x$and p. That is to say,

$$\begin{vmatrix} x_1(t) \\ x_2(t) \end{vmatrix} = \begin{bmatrix} cost & -sint \\ sint & cost \end{bmatrix} \begin{vmatrix} x_1(0) \\ x_2(0) \end{vmatrix}$$

$$\begin{vmatrix} p_1(t) \\ p_2(t) \end{vmatrix} = \begin{bmatrix} cost & -sint \\ sint & cost \end{bmatrix} \begin{vmatrix} p_1(0) \\ p_2(0) \end{vmatrix} \quad (3.2)$$

Angular momentum is a particularly useful concept when a system has rotational symmetry, since in that case the angular momentum is a conserved quantity.(1)

Theorem 3.1: (Stone's Theorem) Suppose $U(\cdot)$ is a strongly continuous one-parameter unitary group on H. Then the infinitesimal generator A of $U(\cdot)$ is densely defined and self-adjoint, and $U(t) = e^{itA}$ for all$t \in R$. If $U(\cdot)$ is a strongly continuous one-parameter unitary

group, then $U(\cdot)$ is continuous in the operator norm topology if and only if the infinitesimal generator of $U(\cdot)$ is a bounded operator .

$$SU\ (2)\left\{\begin{bmatrix}\alpha & -\beta \\ \beta & \alpha\end{bmatrix}, \alpha,\beta \in C, |\alpha|^2, |\beta|^2 = 1\right\} \quad (4)$$

Definition 2.2.

An action of a Lie group G an a manifold M is an assignment to each $g \in G$a diffeomorhismp(G) $\in$ **Diff Msuch that** $p(1) = id\ p(gh) = p(g)p(h)$and such that the map[Moerdijk, 2002 #4}

is a smooth map

Definition2:3 A Lie algebra G is a vector space over a field k with a bilinear composition law$(x;\ y) \to [x;\ y]$

$$[x, ay + bz] = a[x,y] + b[x,z]$$

with $x, y, z \in L$and $a, b \in K$and such that

1. $[x,\ x] = 0$

2. $[x, [y, z]] + [z,\ [x,\ y]] + [y, [z, x]] = 0$; (Jacobi identity)

Notice that

$$[x, y] = [y, x], since\ [x + y, x + y] = 0 = [x, y] + [y, x]$$

3-The Lie Algebra of a Matrix Lie Group

We now associate a Lie algebra g to each matrix Lie group G.

Definition 3.1: If $G \subset GL(n;c)$ is a matrix Lie group, then the Lie

$$g = \{X \in (M_n(c) | e^{tX} \in G\ for\ all\ t \in R\}$$

That is to say, X belongs to g if and only if the one-parameter subgroup generated by X lies entirely in G. Note that to have X belong to g, we need only have e^{tX} belong to G for all real numbers t.

Proposition 3.1: For any matrix Lie group G, the Lie algebra g of G has the following properties.

1. The zero matrix 0 belongs to G.

2. For all X in G, tX belongs to g for all real numbers t.

3. For all X and Y in G, $X + Y$ belongs to g.

4. For all $A \in G$ and $X \in g$ we have $A \times A^{-1} \in g$.

5. For all X and Y in g, the commutator

1-Introduction:

Consider Lie group and Lie algebra have a great role in explaining many concepts in theoretical physics. In this study we trying explain Lie group and Lie algebra application in quantum mechanic

2- Basic concept

If $H$is a Hilbert space and$A \in B(H)$is a non-negative self-adjointoperator on , then it can be shown that A has a well-defined (but possibly infinite) trace. What this means is that the value of ,$trace\ (A) = \sum_j e_i A e_j$is the same for each orthonormal basis$\{e_i\}$of H. Note that since A is a non-negative operator, $e_i e_j$is a non-negative real number, so that the sum is always defined, but may have the value $+\infty$. Now, if A is any bounded operator, then $A^* A$ is self-adjoint and nonnegative. We say that A is Hilbert–Schmidt if

$trace(A^* A) \leq \infty$(1)

Given two Hilbert–Schmidt operators A and B, it can be shown that$A^* B$is a trace-class operator, meaning that the sum

$$trace\ (A) = \sum_j e_i A^* B e_j \qquad (2)$$

Definition 2.1. A Lie group is a set G with two structures: G is a group and G is a (smooth, real) manifold. These structures agree in the following sense: multiplication and inversion are smooth maps.

A morphism of Lie groups is a smooth map which also preserves the group operation:

$$f(gh) = f(g)f(h), f(1) = 1.$$

In a similar way, one defines complex Lie groups. However, unless specified otherwise, "Lie group" means a real Lie group. The following are examples of Lie groups

(1) R^n, with the group operation given by addition

(2)$(R^n, .)$

$(R^+, .)$

(3)$[S^1 = \{z \in c : |z| = 1\}, \ ]$

(4) $GL\ (n, R) \subset R^n$. Many of the groups we will consider will be subgroups of $GL(n, R)$ or $GL(n, c)$

(5) $SU\ (2) = \{A \in GL(n, c) | AA^t = 1, detA = 1\}$. Indeed, one can easily see that

مستخلـص

جبر لى له أهمية كبيره في علم الفيزياء وله استخدامات في الفيزياء الكمية هدفت. الدراسة لتوضيح بعض الجوانب الهندسية لزمر لي وجبر لي، وكذلك إسهاماتها في إنتاج مصفوفات الدورانات المحدودة، كما تهدف إلى معرفة ما إذا كان بالإمكان إستخدام متعدد الطيات التفاضلي لإظهار البنية الهندسية لزمر لي . توصلت الدراسة إلى انه يمكن تمثيل زمر لي وجبر لي هندسيا مستخدمين مفهوم متعدد الطيات التفاضلي وكذلك استنساخ مصفوفات الدورانات المحدودة.

Abstract

The study aims to clarify some geometric aspects of Li groups and Li algebras, as well as their contributions to the production of matrices of finite rotations. The study also aims to use differential manifold to show the geometric structure of Li groups. The study concluded that Li groups and Li algebras can be represented geometrically using sometimes the concept of differential manifold as well as reproducing finite rotation matrices due to the importance of their uses in quantum physics.

Key words: lie group, lie algebra, trace(A), differential manifold.