Geometrical Representation of Lie Group and Lie Algabra

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$\left|\ldots, \pi_{-1}, \pi_0, \pi_1, \ldots\right\rangle$. It represents the juxtaposition (or conjunction, or tensor product) of the number states, ..., $\left|\pi_{-1}\right\rangle \left|,\pi_0\right\rangle$, $\left|\pi_1\right\rangle$,...... located at the individual sites of the lattice (M.V. Karas¯ev,1992) . Recall

$$a\left|n\right\rangle = \sqrt{n}\ |n-1\rangle$$

$$a^{+} n\rangle = \sqrt{n+1}\ |n+1\rangle \text{ for all } n \geq 0, \quad (2.41)$$

While $[a,a^{+}] = 1$

Therefore, it is possible to rely on the previous construction of the effects of creation and annihilation to find formulas that predict the quantum states of the generated particles
Now define a_i so that it applies a to, $\left|\pi_i\right\rangle$. Correspondingly, define a_i^{+}as applying a^{+} to $\left|\pi_i\right\rangle$.

$$\partial_t\ |\pi_1\rangle = -\alpha \sum (2 a_i^{+} a_i - a_{i-1}^{*} a_i - a_{i+1}^{+} a_i)\ |\varphi\rangle$$

$$= -\alpha \sum (a_i^{+} - a_{i-1}^{+})(a_i - a_{i-1}^{+})\ |\varphi\rangle \quad (1.35)\ (2.42)$$

where number state n is replaced by number state n-2 at site i at a certain rate .

thus the state evolves by

$$\partial_t\ |\varphi\rangle = -\alpha \sum (a_i^{+} - a_{i-1}^{+})(a_i - a_{i-1})\ |\varphi\rangle + \lambda \sum (a_i^{2} - a_i^{+2} a_i^{2})\ |\varphi\rangle \quad (2.43)$$

. We denoted by $\emptyset$ the vector space of families $\emptyset = (\emptyset_i)_{i \in I}$ such that $\emptyset_i \in E_i$, consider $\emptyset(x) = \sum_i \langle \varphi_i | \pi_i \rangle_i$ this is implies that for all φ_i, $\pi_i \in E$, the family of numbers $\sum_i \langle \varphi_i | \pi_i \rangle_i$ is
Now from above conception we can make the following generalization:

$$\partial_t\ |\varphi\rangle_j = -\alpha \sum (a_i^{+} - a_{i-1}^{+})(a_i - a_{i-1})\ |\varphi\rangle_j + \lambda \sum (a_i^{2} - a_i^{+2} a_i^{2})\ |\varphi\rangle_j (2.44)$$

$$\partial_t\ |\varphi_\circ\rangle_j = -\alpha \sum (a_i^{+} - a_{i-1}^{+})(a_i - a_{i-1})\ |\varphi_\circ\rangle_j + \lambda \sum (a_i^{2} - a_i^{+2} a_i^{2})\ |\varphi_\circ\rangle_j \quad (2.45)$$

$$\partial_t\ |a_i^{+n} \varphi\rangle_j = -a_i^{n} \sum_{i,j=1}^{n} (a_i^{j} - a_{i-1}^{j-1})(a_i^{+})^{2} - a^{*}{}_{i-1}^{j-1})\ |a^{j} \varphi\rangle_j$$

$$+\lambda_j \sum (a_i^{+})^{j})^{2} - (a_i^{+j})^{2} a_i^{j^2})\ |a^{+n} \varphi\rangle_j \qquad (2.46)$$

The equations (2.44) , (2.45) and (2.46) are a mathematical formulas that represents the future of wavefunction space when repeating creation and annihilation operators.

$$a^{+} = \frac{1}{\sqrt{2}}(\tilde{x} - \frac{d}{d\tilde{x}}) \qquad (2.36)$$

Note that the constants m, ω, and ℏ have conveniently disappeared from the formulas.

Given the expression in (2..36)), we can easily solve the (first-order, linear) equation $a\varphi_n = 0$ as

$$\varphi_{\circ}(\tilde{x}) = Ce^{-\tilde{x}/2} \qquad (2.37)$$

If we take C to be positive, then our normalization condition determines its value to be $\sqrt{\pi/D}$.

Obtain, then,

$$\varphi_{\circ}(x) = \sqrt{\frac{\pi m\omega}{\hbar}} \exp\{-\frac{m\omega}{\hbar}x^2\} \qquad (2.38)$$

It remains only to apply a^{+}repeatedly to $\varphi_{\circ}$ to get the "excited states" φ_n

Theorem 11.3 The ground state $\varphi_{\circ}$ of the harmonic oscillator is given by (2.37). The excited states φ_n are given by

$$\varphi_n = H_n \varphi_{\circ} \qquad (2.39)$$

Where H_n is a polynomial of degree n given inductively by the formulas?

$H_{\circ}(\tilde{x}) = 1$

$$H_{n+1}(\tilde{x}) = \frac{1}{\sqrt{2}}(2\tilde{x}\, H_n(\tilde{x}) - \frac{d\, H_n(\tilde{x})}{d\, \tilde{x}})$$

Here, $\tilde{x}$ is the normalized position variable given by(2.34)

Proof.

When n = 0,by (2.34), reduces to $\varphi_{\circ} = \varphi_{\circ}$. Assuming that (2.39) holds for some n, we compute φ_{n+1} as

$$\varphi_{n+1} = a^{+}\ \varphi_n = \frac{1}{\sqrt{2}}(\tilde{x}\, H_n(\tilde{x})\, C\, e^{-\frac{\tilde{x}^2}{2}} - \frac{d}{d\tilde{x}}[H_n(\tilde{x})\, C\, e^{-\frac{\tilde{x}^2}{2}}] \qquad (2.39)$$

$$= \frac{1}{\sqrt{2}}(\tilde{x}\, H_n(\tilde{x}) \frac{d\, H_n}{d\tilde{x}})\, Ce^{-\frac{\tilde{x}^2}{2}} = H_{n+1}(\tilde{x})\varphi_{\circ}(\tilde{x}) \qquad (2.40)\ (\text{I.E. Segal, 1993})$$

Now we can describe the occupation of particles on the lattice as a [ket] of form :

Since $a^{*}a$ cannot have negative eigenvalues, we may call φ_0 a "ground state" for $a^{*}a$, that is, an eigenvector with lowest possible eigenvalue. We may then apply the raising operator a^{*} repeatedly to φ_0 to obtain eigenvectors for $a^{*}a$ with positive eigenvalues.

Theorem 1.1 If φ_0 is a unit vector with the property that $a\varphi_0 = 0$, then the vectors

$\varphi_n = (a^{+})^{n}\varphi_0$, $n \geq 0$

Satisfy the following relations for all n, m ≥ 0:

$$a^{+}\varphi_n = \varphi_{n+1} \qquad (2.32)$$

$$a^{+}a\varphi_n = n\,\varphi_n \qquad (2.33)$$

$$(\varphi_n , \varphi_m) = n!\,\delta_{m,n}$$

$$a\varphi_{n+1} = (n+1)\varphi_n \qquad (2.34)$$

Let us think for a moment about what this is saying. We have an orthogonal "chain" of eigenvectors for $a^{*}a$ with eigenvalues 0, 1, 2, ..., with the norm of φ_n equal to $\sqrt{n!}$. The raising operator a^{+} shifts us up the chain, while the lowering operator a shifts us down the chain (up to a constant). In particular, the "ground state" φ_0 is annihilated by a. Thus, we have a complete understanding of how a and a^{+} act on this chain of eigenvectors for a^{+}a.

Proof.

The first result is the definition of φ_{n+1} and the second follows from Proposition 1.1 and the fact that $a^{+}a\varphi_0 = 0$. For the third result, if n = m, we use the general result that eigenvectors for a self-adjoint operator (in our case, $a^{+}a$) with distinct eigenvalues are orthogonal. (This result actually applies to operators that are only symmetric.) If n = m, we work by induction. For n = 0, $(\varphi_0, \varphi_0) = 1$ is assumed. If we assume $(\varphi_n, \varphi_m) = n!$, we compute that

$$(\varphi_{n+1}, \varphi_{n+1}) = (a^{+}\varphi_n , a^{+}\varphi_n) = (\varphi_n , aa^{+}\varphi_n)$$

$$= (\varphi_n , a^{+}a + 1)\varphi_n)$$

$$= (n+1)\,(\varphi_n , \varphi_n)$$

$$= (n+1)! \quad \text{[Am2009]}$$

Finally, we compute that

$$a\varphi_{n+1} = aa^{+}\varphi_n = (aa^{+} + 1)\varphi_n = (n+1)\varphi_n \quad (2.35)$$

A calculation gives the following simple expressions for the raising and lowering operators:

$$a = \frac{1}{\sqrt{2}}\left(\bar{x} + \frac{d}{d\bar{x}}\right)$$

Proposition 2.1

Suppose that φ is an eigenvector for $a\, a^{+}$ with eigenvalue λ. Then

$$a^{+}a\,(a\varphi) = (\lambda - 1)\, a\varphi \qquad (2.29)$$

$$a^{+}a\,(a^{+}\varphi) = (\lambda + 1)\, a^{+}\varphi. \qquad (2.30)$$

Thus, either $a\varphi$ is zero or $a\varphi$ is an eigenvector for $a^{+}a$ with eigenvalue λ - 1. Similarly, either $a^{+}\varphi$ is zero or $a^{+}\varphi$ is an eigenvector for $a^{+}a$ with eigenvalue λ + 1. That is say, the operators a^{*} *and* a raise and lower the eigenvalues of $a^{*}a$, respectively.[(McGraw-Hill, New York, 1991)]

Proof

Using the commutation relation (2.29) we find that

$$a^{+}a\,(a\varphi) = a\,(a^{+}a\,) - a)\,\varphi) = (\lambda - 1)\, a\varphi$$

A similar calculation applies to $a^{+}\varphi$, using (2.30) .

If φ is an eigenvector for $a^{+}a$ with eigenvalue λ, then

$\lambda(\varphi,\varphi) = (\varphi, a^{+}a\varphi) = (a\varphi, a\varphi) \geq 0$

which means that λ ≥ 0. Let us assume that $a^{+}a$ has at least one eigenvectorφ, with eigenvalue λ, which we expect since $a^{*}a$ is self-adjoint. Since a lowers the eigenvalue of $a^{+}a$, if we apply a repeatedly to φ, we must eventually get zero. After all, if an φ were always nonzero, these vectors would be, for large n, eigenvectors for $a^{*}a$with negative eigenvalue, which we have seen is impossible.

It follows that there exists some N ≥ 0 such that $a^{N}\varphi \neq 0$ but $a^{N+1}\varphi = 0$. If we define φ_{*} by

$\varphi_{*} = a^{N}\varphi$ (2.31)

then aφ_{*} = 0, which means that $a^{*}a\varphi_{*}$ = 0. Thus, φ_{*} is an eigenvector for $a^{*}a$ with eigenvalue 0. (It follows that the original eigenvalue λ must have been equal to the non-negative integer N.) The conclusion is this: Provided that $a^{*}a$ has at least one eigenvector φ, we can find a nonzero vector φ_{*} such that

$$a\varphi_{*} = a^{+}a\varphi_{*} = 0$$

$$\Rightarrow [\emptyset(x), \pi(x)] = \int \frac{d^3p\, d^3_{p\backslash}}{(2\pi)^6} \times -\frac{i}{2} \int \frac{\omega\backslash}{wp}$$

$$\left[a_{-p}, a^{+}_{p\backslash}\right] \quad \left[a_p, a^{+}_{-p\backslash}\right] \quad e^{i[px+p\backslash x\backslash]} - i\delta^3(x-x^1) \quad (2.21)$$

Using the equations (2.10) , (2.11) and 2,3 from Dirac delta functions

$$\left[a_p, a^{+}_{-p\backslash}\right] - 2\pi^3\ \delta^3(p-p\backslash)$$

Commutation relation for creation and scalar field

Using equation **(2.21)**

$$\mathcal{H} - \int d^3x\, \mathcal{H} -$$

$$\int d^3x\, [-\frac{1}{2}\pi^2 + \frac{1}{2}(m\emptyset)^2 + \frac{1}{2}(\nabla\emptyset)^2 \quad (2.22)$$

$$\mathcal{H} = \int d^3x \int \frac{d^3p\, d^3_{p\backslash}}{(\pi\pi)^6} \times e^{i[p+p\backslash]x} [-\frac{\sqrt{\omega_p\omega_p^\backslash}}{4} \quad (2.23)$$

$$\left[a_p, -a^{+}_{-p\backslash}\right] \quad \left[a_{p\backslash}, -a^{+}_{p\backslash}\right] \quad + \frac{-pp\backslash + m^2}{4\sqrt{\omega_p\omega_p^\backslash}} \left[a_p, +a^{+}_{-p\backslash}\right] \quad \left[a_{-p\backslash}, +a^{+}_{p\backslash}\right] \quad (2.24)$$

$$-(2\pi)^3 \int \frac{d^3p\, d^3_{p\backslash}}{(\pi\pi)^6} \times \delta(p+p\backslash)[-\frac{\sqrt{\omega_p\omega_p^\backslash}}{-1}[a_p, -a_p^{+}] \left[a_p, +a^{+}_{-p\backslash}\right] \frac{-pp\backslash + m^2}{\sqrt{\omega_p\omega_p}} \left[a_p, -a_p^{+}\right]$$

$$\left[a_p, -a^{+}_{p\backslash}\right] (2.24) \quad -\int \frac{d^3p\, d^3_{p\backslash}}{(\pi\pi)^6} \frac{-\omega_p}{4} \left[a_p, -a^{+}_{-p}\right] \left[a_{-p}, -a^{+}_{-p\backslash}\right] \quad (2.25)$$

$$\mathcal{H} = \int \frac{d^3p}{(\pi\pi)^3} \omega_p [a_p a_p^{+} + \frac{1}{2}[a_p, a_p^{+}] \quad (2.26)$$

s.t $\frac{1}{2}[a_p, a_p^{+}]$ is vaccum state

$$[a_p, a_p^{+}] - \mid \delta^3(0) \quad (2.27)$$

$[\mathcal{H}, a^{+}] - \omega_p a_p^{+}$ creation of particles [quantize]

$$[\mathcal{H}, a] = -\omega_p a_p \quad (2.28)$$

In equations **(2.27)** and **(2.28)** we obtained to quantize of quantum field

Use:

This is scalar field equation .

We how that $(\Box^2+ m^2)\, \emptyset\,(t,\ x) = o$ **[Ann. Inst. H. Poincar´e Phys. Th´eor59, 357–381 (1993)]** (2.16)

And it is not harmonic oscillator , must make fore this clearly harmonic oscillator and spectrum

$$(\Box^2+ m^2)\, \emptyset\,(t,x) - \int \frac{d^3}{(2\pi)^3} [(\Box^2+ m^2)\, \emptyset\,(t,x)\, [e^{ip_ix}\ \ \emptyset\,(p,t)]$$

$$- \int \frac{d^3p}{(2\pi)^3} \left[\frac{d^2}{dx^2}\ |p|^2 + m^2\right] \emptyset\,(p,t) = o$$

$$- \left[\frac{d^2}{dx^2} + [|p|^2] + m^2\right] \emptyset(p,t) = o \qquad (2.17)$$

This is harmonic oscillator fore Klein Gordon equation

$\omega_p - \sqrt{p^n + m^n}$

now we can write Harmonic oscillator for structure of [Q.F]

We use $\frac{d^3p}{(2\pi)^3} \times \frac{1}{\sqrt{2\omega p}}$ lorentes inverient

Element in G F T [scalar field]

$$\emptyset(x) - \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p e^{ip_ix} + a^+ e^{-ip_ix}\ \right] \frac{d^3p}{(2\pi)^3} \qquad (2.18)$$

$$\pi\,[x] = \int \frac{d^3p}{(2\pi)^3} \quad (-1) \int \sqrt{\frac{\omega_p}{2}} \left[a_p e^{ip_ix} - a^+ e^{-ip_ix}\ \right] \qquad (2.19)$$

And use

$$\left(\frac{d^2}{dt^2} + p^2 + m^2\right) \emptyset(p,t) = 0 \qquad (2.20)$$

Using $[\emptyset\,(x), \pi\,(x)] = i\delta^3\left(x - x^{\backslash}\right)$

5) $\delta\,(a\,x) = \ |a\,|^{-n}\delta\,(\,x)$

$\delta^{\backslash}\,(-x)\ =\ -\delta^{\backslash}\,(x)$

$x\,\delta^{\backslash}\ (x) = \ -\delta^{\backslash}\,(x)$

2. Quantization of scalar field

consider the particle (boson has spin – 0)

In historical in quantum mechanics

$$([q_j\ p_j] = \ \delta_{ij}\quad (1.1) \qquad\qquad (2.1)$$

$$[p_j\,,p_j] = 0\ ;\ [q_i\,,q_j] = 0 \qquad\qquad (2.2)$$

Now we transfer this idea to use in constructing the quantum field [Q . F]

$q_j\ \rightarrow\ \varphi\,(x)\quad p_j\ \rightarrow\ \pi\,(x)$

Such that

$[\varphi(x)\,,\pi\,(x)\,] = i\,\delta^3\ (\ (x - x^{\backslash})$ (2.3)

S.t $\delta^3\ (x - x^{\backslash})$ is Dirac delta function

When performing the quantization process must be calculation spectrum field

By generating the state using oscillator harmonic:

Start for Fourier transform

$$\emptyset\,(x,t) = \ \int \frac{d^3p}{(2\pi)^3}\quad e^{ipx}\ \ \emptyset\,(p,t) \qquad\qquad (2.4)$$

And from solar field equation

$$[H_{SHO}\ \ ,a^{+}\,] = \ -\ \omega\,a^{+} \qquad\qquad (2.5)$$

$$|\,n\rangle = \ (a^{+})^{n}\ |o\rangle \qquad\qquad (2.6)$$

To build the quantization process for the quantum field, we now use the same method, namely, the creation and annihilation operators.

مستخلـص

هدف الدراسة استخدام مؤثرات التخلق و الإفناء في فهم البناء الرياضي لنظرية الحقل الكمومي وعملية تكميم الحقل الكمي كما تهدف لإيجاد صيغ رياضية لفضاء الدالة الموجية عند تكرار هذه المؤثرات . توصلت الدراسة إلى بناء عملية التكميم في الحقل الكمومي باستخدام مؤثرات التخلق والإفناء بصورة رياضية مبسطة من اجل تبسيط الفهم الرياضي لنظرية الحقل الكمومي.كما توصلت الدراسة لإيجاد صيغة رياضية لمستقبل فضاء الدالة الموجية عند تكرار مؤثرات التخلق والإفناء.

Abstract

The aim of the study is to use the creation and annihilation operators in understanding the mathematical structure of the quantum field theory and the process of quantizing the quantum field. It also aims to find mathematical formulas for the wave function space when these operators are repeated. Results, to construct the quantization process in the quantum field using the creation and annihilation operations in a simplified mathematical way in order to simplify the mathematical understanding of the of the quantum field theory. The study also obtained to find a mathematical formula for the future of the wave function space when the effects of creation and annihilation are repeated.