# Lie Group and Spin Representation
# (Representation)

## (تمثيل زمرة لِي وزمرة اللف المغزلي)

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**المستخلص**

الهدف من هذه الورقة العلمية توضيح تمثيل زمرة لِي وزمرة اللف المغزلي ، ثم توضيح الدوران في الفضاءات النونية مستحدثةً زمرة معينة ، مُثل جبر لِي براسم محتفظاً بخصائص أقواس لِي. وعمم التمثيل علي المصفوفات في الفضاء المركب حيث الزمرة الموحدة في الفضاء الثنائي والزمرة الموحدة الخاصة في الفضاء الثلاثي . وتم التعبير في فترات أوسع عن زمرة اللف المغزلي في الفضاء النوني .

**Abstract**

The main goal of this paper is to discuss the representation theory, to explain how rotations in $R^n$ space are induced by the action of a certain group, a Lie algebra representation as map of Lie algebras preserving the Lie bracket. In way that generalizes the action of the unit complex numbers unitary group, on $R^2$ and special unitary group, on $R^3$. We expressed in terms of multiplication in a large algebra containing both the group spin(n) and $R^n$.

**Introduction:-**

A (complex) representation $(\pi, V)$ of a group $G$ on a complex vector space $V$ with chosen basis identifying $V \cong C^n$) is a homomorphism

$$\pi: G \to GL(n, C)$$

This basically a set of $n$ $by$ $n$ matrices, one for each group element, satisfy the multiplication rules of the group elements, $n$ is called the dimension of the representation. The groups $G$ we are interested in will be examples of what mathematicians call " Lie group".

For a representation $\pi$ and group elements $g$ that are close to the identity, one can use exponentiation to write $\pi(g) \in GL(n, C)$as

$$\pi(g) = e^A$$

where A is also a matrix, close to zero matrix.

Given representations $\pi_1$ and $\pi_2$ of dimensions $n_1$, and $n_2$, one can define another representation of dimensions $n_1 + n_2$ called direct sum of the two representations, denoted by $\pi_1 \oplus \pi_2$ this representation is given by the homomorphism

$$(\pi_1 \oplus \pi_2): g \in G \to \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix}$$

In other words, one just takes as representations matrices block-diagonal matrices with $\pi_1$ and $\pi_2$ giving the blocks.

To understand the representations of a group G one proceeds by first identifying the irreducible ones, those that cannot be decomposed into two representations of lower dimension.

A representation $\pi$ is called irreducible if it cannot be put in the form $\pi_1 \oplus \pi_2$, for $\pi_1$ and $\pi_2$ of dimension greater than zero.

**[1] The Group $U(1)$and its Representations:-**

The elements of group $U(1)$ are points on the unit circle, which can be label by unit complex number $e^{i\theta}$, for $\theta \in R$ .Note that $\theta$ *and* $\theta + 2\pi n$ label the same group element: Multiplication of group elements is just complex multiplication which by properties of exponential satisfies $e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$.

- So in terms of angle θ the group law is just addition $(mod\ 2\pi)$.
- Since $U(1)$ is commutative group, all irreducible representation will be one-dimensional.
- such an irreducible representation will be given by a map

$$\pi: U(1) \rightarrow GL(1, C)$$

But an invertible 1 by1 matrix is just an invertible complex number, and will denote of these as $C^*$.

**Theorem(1-1):-**

All irreducible representations of the group $U(1)$are unitary , and given by

$\pi_k : \theta \in U(1) \rightarrow \pi_k(\theta) = e^{ik\theta} \in U(1) \subset GL(1, C) \cong C^*$ for k$\in Z$.

**Proof:-**

The given $\pi_k$ satisfy the homomorphism property

$$\pi_k(\theta_1 + \theta_2) = \pi_k(\theta_1)\pi_k(\theta_2)$$

And periodicity property $\pi_k(2k) = \pi_k(0) = 1$.

We just need to show that any $f: U(1) \rightarrow C^*$ satisfying the homomorphism and periodicity properties is of this form.

Computing the derivative $f'(\theta) = \frac{df}{d\theta}$ we find

$f'(\theta) = \lim_{\delta\theta \rightarrow 0} \frac{f(\theta+\delta\theta)-f(\theta)}{\delta\theta}$ (Using the homomorphism property)

$= f(\theta) \lim_{\delta\theta \rightarrow 0} \frac{f((\delta\theta)-1)}{\delta\theta}$

$=f(\theta)f'(0)$

Denoting the constant $f'(0)$ by $C$, the only solution to this differential equation satisfying $f(0) = 1$ are $f(\theta)=e^{C\theta}$

Requiring periodicity we find $f(2\pi) = e^{C2\pi} = f(0) = 1.$

Which implies $C = ik$ for $k \in Z$, and $f=\pi_k$ for some integral $k$.

The representation we have found are all unitary, with $\pi_k$ taking values not just in $C^*$, but in $U(1) \subset C^*$.

**1-1 The Charge Operator:-**

The general principle that since the state space $H$ is a unitary representation of Lie group, we get an associated self- adjoint operator on $H$.

For the case of $G = U(1)$, this operator is just the operator that acts by multiplication by the integer q on the representation space $C$ of the irreducible representation $(\pi_q , G)$ .Since the irreducible representation of $G = U(1)$ are all one- dimensional , this means that as a $U(1)$ representation, we have

$$H = H_{q_1} + H_{q_2} + \cdots + H_{q_n}$$

For some set of integers $q_1, q_2, \ldots, q_n$ ( $n$ is dim of $H$, the $q_i$ may not be distinct ). We will call this operator the charge operator.

**Definition(1-1):-**

The charge operator $Q$ is the self adjoint linear operator on $H$ that acts by multiplication by $q_i$ on the irreducible representation $H_{q_i}$ .It acts on $H$ as the matrix

$$\begin{bmatrix} q_1 & 0 & \ldots & 0 \\ 0 & q_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \cdots & q_n \end{bmatrix}$$

$Q$ is the quantum mechanical observable , operator on $H$ . From the action of G on $H$, one can recover the representation, i.e., the action of symmetry group $U(1) on H$ , by multiplying i and exponentiation to get

$$\pi(\theta) = e^{iQ\theta} = \begin{pmatrix} e^{iq_1\theta} & o & o..........o \\ o & e^{iq_2\theta} & .............o \\ ........................... \\ ........................... \\ o & o................e^{iq_n\theta} \end{pmatrix} \in U(n) \subset GL(n,C)$$

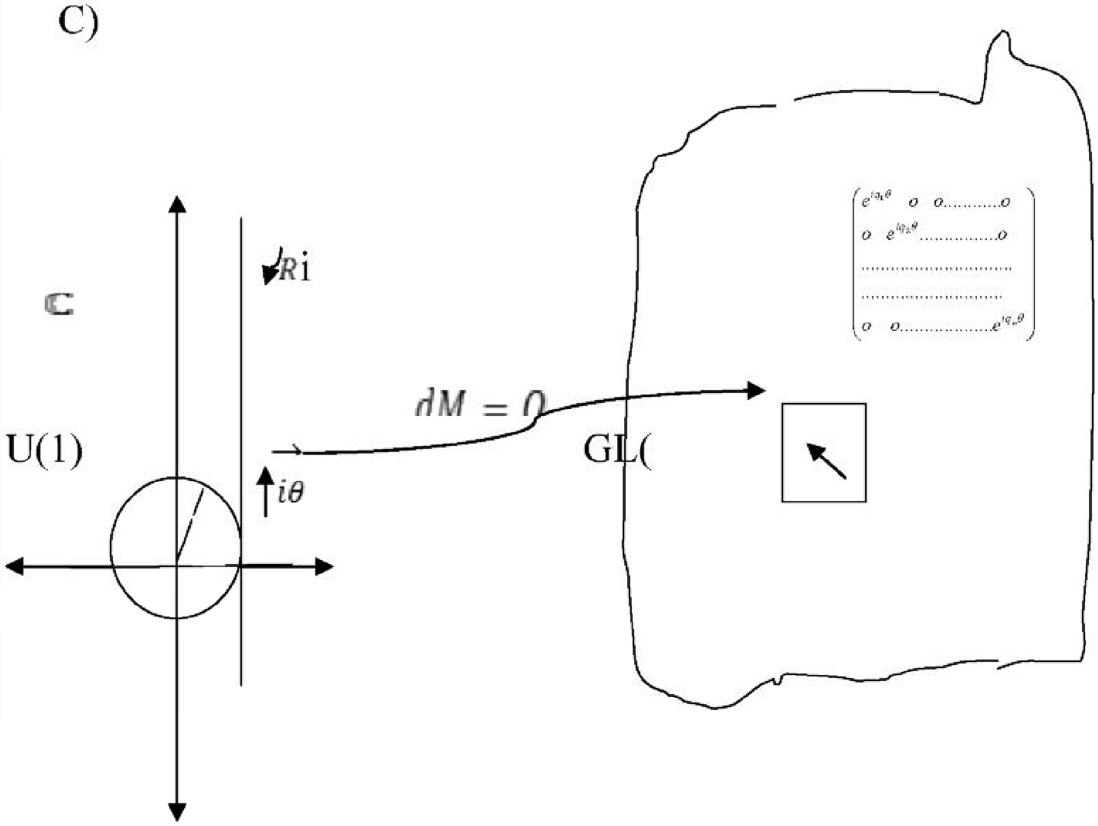
(2.4)

The standard physics terminology is that "Q generates the $U(1)$ symmetry transformation".

The general abstract high – powered mathematical point of view is that the representation π is a map between manifolds , from the Lie group $U(1)$ to the Lie group $GL(n,C)$, that takes identity of $U(1)$ to the identity space of $GL(n,C)$. As such it has differential, $\pi'$, which is a map from tangent space at the identity of $U(1)$( which here is $iR$) to the tangent space to identity of $GL(n,C)$ which is the space $M(n,C)$, the n by n complex matrices. The tangent space at the identity of a Lie group is called "Lie algebra". Here the relation between the differential of $\pi$ and the operator $Q$ is

$$\pi' : i\theta \in iR \to \pi'(i\theta) = iQ\theta$$

One can sketch the situation as follows: GL(n, C)




Fig(1): **The representation $\pi$ is a map between manifolds , from the Lie group $U(1)$ to the Lie group $GL(n, C)$.**

## 1-2 Dual Spaces and Inner Products:-

Given a vector space $V$ over afield K, dual vector space $V^*$ is the set of all Linear map $V \to K$ $i.e.,$:

$V^* = \{L: V \to K \ such\ that\ L(\alpha v + \beta w) = \alpha L(v) + \beta L(w)\}$

For $\alpha, \beta \in K, v, w \in V$. An e element of a vector space $V$ is written as "ket vector ".

$|v\rangle$ where $v$ is a label for a vector. An element of the dual vector space $V^*$ is written as a" bra vector" $\langle l|$

-Evaluating $l \in V^*$ on $v \in V$ gives an element of $K$ , written $\langle l|v\rangle$.

If $\Omega: V \rightarrow V$ is linear map

$$\langle l|\Omega|v\rangle = (l|\Omega\, v) = l(\Omega\, v)$$

**Definition(1-2):-** ( Inner Product, Real case)

An inner product on real vector space V is map $\langle .,.\rangle: V\times V \rightarrow R$.

That is linear in both variables and symmetric $(\langle v,\omega\rangle = \langle \omega, v\rangle)$.

**Definition(1-3):-** (Inner Product Complex case)

An Hermitic inner product on complex vector space $V$ is map $\langle .,.\rangle: V\times V \rightarrow C$ That is linear in the second variables and ,anti linear in the first variablesi.e, for $\alpha,\beta \in C$ *and* $u, v, w \in V$.

$$\langle \alpha u + \beta v, w\rangle = \overline{\alpha}\langle u,w\rangle + \overline{\beta}\langle v, w\rangle$$

And conjugate symmetric $\langle v, w\rangle = \overline{\langle w, v\rangle}$.

**[2] Bases, Linear Operators and Matrix Elements:-**

In particular an orthonormal basis $\{e_i\}$, $i = 1 \ldots n$ satisfy $<e_i, e_j> = \delta_{ij}$. We will denote basis vectors in the bra-ket notation by $|i\rangle = e_i$ .

An arbitrary vector $v \in V$ can be expressed as

$$v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n$$

The linear function on $V$ which takes value the coefficient $v_i$of the basis vector $e_i$ on a vector $v \in V$ , $v_i = \langle e_i, v\rangle$

In bra-ket notation we have $v_i = \langle e_i|v\rangle$ and $|v\rangle = \sum_{i=1}^{n} |i\rangle\langle i|v\rangle$

For corresponding elements of $V^*$, one has (using anti-linearity)

$\langle v| = \langle v| = \sum_{i=1}^{n} \overline{v_i}\langle i| = \sum_{i=1}^{n}\langle v|i\rangle\langle v|$

The operation of taking a vector $|v\rangle$to a dual vector $\langle v|$ corresponds to taking a column vector to the row vector that it is conjugate- transpose. $\langle v| = (\overline{v_1}, \overline{v_2}, \ldots, \overline{v_n})$ .Then one has

$$\langle v|\omega\rangle = (\overline{v_1}, \overline{v_2}, \ldots, \overline{v_n})\begin{bmatrix}\omega_1\\ \vdots \\ \omega_n\end{bmatrix} = \overline{v_1}\omega_1, \overline{v_2}\omega_2, \ldots, \overline{v_n}\omega_n$$

If $\Omega$is a linear operator $\Omega: V \to V$ , with matrix elements $\Omega_{ij}: < j|\Omega_i >$. As matrices the action of $\Omega$ on $|v\rangle$given by:

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \to \begin{bmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \vdots & \vdots & \cdots & \vdots \\ \Omega_{n1} & \Omega_{n2} & \cdots & \Omega_{nn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

The decomposition of a vector $\upsilon$ in terms of coefficients $|v\rangle = \sum_{i=1}^{n} |i\rangle\langle i|v\rangle$.

**2-1 Adjoint Operators :**

The adjoint of a linear operator $\Omega: V \to V$ is operator $\Omega^{\dagger}$ satisfying $< \Omega\upsilon , \omega > \; = \; < \upsilon, \Omega^{\dagger}\omega >$ or in bar-ket notation

$$< \Omega\upsilon \mid \omega > \; = < \upsilon \mid \Omega^{\dagger}\omega > \; \forall v, \omega \in V$$

Generalizing the fact $< \alpha v \mid = \bar{\alpha} < v|, for\ \alpha \in C, \; < \Omega v \mid \; =< v \mid \Omega^{\dagger}$. $<\Omega v|$ is conjugate transposed of $|\Omega\, v\,\rangle$.

$(\Omega^{\dagger})_{ij}=\overline{\Omega_{ij}}$ , the linear transformation is self adjoint if $\Omega^{\dagger}= \Omega$ , skew – adjoint if $\Omega^{\dagger}= -\Omega$ .

**2-2 Orthogonal and unitary transformations:-**

A special class of linear transformations will be Invertible transformations that preserve the inner product, i.e., satisfying

$$< \Omega\upsilon, \Omega\omega > \; =< \Omega\upsilon \mid \Omega\omega > = < \upsilon, \omega > = < \upsilon| \; \omega >$$

$\forall\, v, \omega \in V$. Such transformations take orthonormal bases to orthonormal bases

$$< \Omega\upsilon \, , \Omega\omega > = < \upsilon \, , \Omega^{\dagger}\Omega\omega > \; = < \upsilon \, , \omega >$$

So $\Omega^{\dagger}\Omega = 1$ or equivalently $\Omega^{\dagger}= \Omega^{-1}$ .

In matrix rotating, this first condition becomes:

$$\sum_{j=1}^{n} (\Omega^{\dagger})_{ij}\, \Omega_{jk} = \sum_{j=1}^{n} \bar{\Omega}_{ij}\, \Omega_{jk} = \delta_{ik}$$

Which says that the column vectors of matrix for $\Omega$ are orthonormal vectors.

## 2-3 Orthonormal Groups

The orthonormal groups $O\ (n)$ in $n$-dimensional is the group of invertible transformations preserving inner product on a real $n$-dimensional vector space V.

This is also the group of n by n real invertible matrices satisfying

$$(\Omega^{-1})_{ij} = \Omega_{ji}$$

The sub group of $O(n)$ of matrices with determinant 1 (equivalently , the sub group preserving orientation of orthonormal bases) is called $SO(n)$.

We have

$\Omega^{-1}\ \Omega = 1 \Rightarrow \det\ (\Omega^{-1})\ \det\ (\Omega) = \det\ (\Omega^{\dagger})\ \det\ (\Omega) = (\det\Omega)^2 = 1$ .

So, $\det\ (\Omega) = \pm\ 1$ .

$O(n)$ is continuous Lie group , with two components:

$SO(n)$, the subgroup orientation preserving transformations all elements of $SO(\ 2)$, are given by matrices of the form:

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

These matrices given counter -clock wise rotations in $R^2$ by angle θ. The other component of $O(2)$ will be given by matrices the form

$$\begin{pmatrix} cos\theta & sin\theta \\ sin\theta & -cos\theta \end{pmatrix}$$

Note that the group $SO(\ 2)$ is isomorphic to group $U(1)$ by

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \Leftrightarrow e^{i\theta}$$

**Definition(2-1):-** (Unitary Groups)

The unitary groups $U(n)$ in $n$-dimension is the group of invertible transformations preserving a Hermitian product on a complex $n$- dimensional vector space $V$. This is also the group of $n\ by\ n$ complex invertible matrices satisfying

$$(\Omega^{-1})_{ij} = \overline{\Omega_{ji}} = (\Omega^{\dagger})_{ij}$$

The sub group of $U(n)$ of matrices with determinant 1 is called SU (n).

The Same calculation as in the real case here gives

det $(\Omega^{-1})$ det $(\Omega)$ =det$(\Omega^{\dagger})$det$(\Omega)$= $\overline{\det(\Omega)}$ det $(\Omega)$= $|det\ (\Omega)|^2 = 1$.

So det (Ω) is a complex number of length one. The map

$$\Omega \in U(n) \rightarrow det\ (\Omega) \in U(1)$$

Is a group homomorphism .

**[3] Lie Algebras and Lie Algebras Representations :**

For a group G we have defined unitary representations $(\pi, V)$ for finite dimensional vector space of complex dimension n as homomorphism $\pi: G \rightarrow U(n)$

**Definition(3-1):-** (Lie Algebra)

For $G$ Lie group of $n$ *by* $n$ invertible matrices, the Lie algebra of $G$ (written Lie($G$) or $\mathcal{G}$) is the space of n by n matrices $X$ such that $e^{tX} \in G$ for $t \in R$.

**Definition(3-2):-** (Adjoin Representation )

The adjoin representation (Ad, $\mathcal{G}$) is given by H the homomorphism Ad: $\mathcal{G} \in$G→ {x→ $\mathcal{G}\, x\, \mathcal{G}^{-1}$} ∈GL($\mathcal{G}$) meaning

$$(Ad\ (\mathcal{G}))x = \mathcal{G} \times \mathcal{G}^{-1}$$

To show that this is well defined, one needs to check that $\mathcal{G}\, x\, \mathcal{G}^{-1} \in \mathcal{G}$ when x$\in \mathcal{G}$ , but this can be shown using the identity

$$e^{t\mathcal{G}\, x\, \mathcal{G}^{-1}} = \mathcal{G}\, e^{tx}\, \mathcal{G}^{-1} \Rightarrow e^{t\mathcal{G}\, x\, \mathcal{G}^{-1}} \in G\ if\ e^{tx} \in G$$

To check this, just expand the exponential and use

$$(\mathcal{G}x\mathcal{G}^{-1})^k = (\mathcal{G}x\mathcal{G}^{-1})(\mathcal{G}x\mathcal{G}^{-1}) \ldots (\mathcal{G}x\mathcal{G}^{-1}) = \mathcal{G}x^k\mathcal{G}^{-1}$$

It is homomorphism, with

$$\mathrm{Ad}(\mathcal{G}_1)\ \mathrm{Ad}(\mathcal{G}_2) = \mathrm{Ad}(\mathcal{G}_1\mathcal{G}_2)$$

The adjoint representation $(\mathrm{Ad}, \mathcal{G})$ is in general not complex representation , but a real with $\mathrm{Ad}(\mathcal{G}) \in \mathrm{GL}(\mathcal{G}) = \mathrm{GL}(\dim \mathcal{G}, \mathrm{R})$.

**Definition(3-3):-** (Lie bracket)

The Lie bracket operation on $\mathcal{G}$ is the bilinear anti-symmetric map given by the commutator of matrices

$$[\,.\,,.\,] : (X,Y) \in \mathcal{G} \times \mathcal{G} \rightarrow [\,X,Y] = XY - YX \in \mathcal{G}$$

**Theorem(3-1):-**

if $X\,,Y \in \mathcal{G}\,,[\,X,Y] = XY - YX \in \mathcal{G}$

**Proof:**

Since $X \in \mathcal{G}$ , have $e^{tX} \in \mathrm{G}$ , $y \in \mathcal{G}$ , we have $e^{tY} \in \mathrm{G}$ by the adjoint representation

$$Ad(e^{tX})_Y = e^{tX} Y e^{-tX} \in \mathcal{G}$$

As t varies this gives us parameterized curve in g. It is velocity vector will also be in g, so. $\frac{d}{dt}(e^{tX} Y e^{-tX}) \in \mathcal{G}$

$$\frac{d}{dt}(e^{tX} Y e^{-tX}) = \frac{d}{dt}(e^{tX} Y))e^{-tX} + e^{tX} Y \left(\frac{d}{dt}(e^{-tX})\right)$$

$$= X e^{tX} Y e^{-tX} - e^{tX} Y X e^{-tX}$$

Evaluating this at t= o gives

$XY - YX$

Which is thus shown to be in $\mathcal{G}$ .

To do calculations with a Lie algebra choose basis $X_1, X_2, \dots, X_n$ for vector space g , the Lie bracket can be written in terms of this basis as $[X_j, X_k] = \sum_{l=1}^{n} C_{jkl} X_l$

**3-1 Lie Algebra of the Orthogonal and Unitary Groups:-**

The groups we are most interested in, are the groups of linear

Transformations preserving an inner product the orthogonal and unitary groups. Subgroups of $GL(n,R)$ *or* $GL\ (n,C)$ of elements $\Omega$ satisfying the conditions $\Omega\Omega^{\dagger} = 1$.

In order to see what this condition Becomes on the Lie algebra, write $\Omega = e^{tX}$, for some parameter $t, X$ matrix in the Lie algebra matrices is since transpose of a product matrices is product of transpose matrices .

i.e., $(XY)^T = Y^T X^T$

And the complex conjugate of product of the matrices is the product complex conjugates of matrices, one has:

$$(e^{tX})^{\dagger} = e^{tX^{\dagger}}$$

The condition $\Omega\Omega^{\dagger} = 1$

Thus becomes $e^{tX}(e^{tX})^{\dagger} = e^{tX}e^{tX^{\dagger}} = 1$

Since X and$X^{\dagger}$ commute, this becomes $e^{t(X+X^{\dagger})} = 1$ or $X + X^{\dagger} = 0$

So the matrices we want to exponent are skew- adjoint , satisfying $X^{\dagger} = -X$.

**3-2 Lie Algebra representations:**

A (Complex) Lie algebra representations $(\phi, V)$ of Lie algebra $\mathcal{G}$ on an $n - dimensional$ complex vector space $V$ is given by a linear map.

$$\phi : X \in \mathcal{G} \rightarrow \phi\ (X) \in gL\ (n,C) = M(n,C)$$

Satisfying $\phi$ ([$X,Y$] = [ $\phi$ (X) , $\phi$ ($Y$) ] such a presentation is called unitary if its image is in $U(n)$, i.e., is satisfies

$$\phi(X)^{\dagger} = -\phi\ (X)$$

Given a basis $X_1\ , X_2, \dots, X_d$ of Lie algebra of dimension d with structure constant $C_{jk}$ a representation given by choice of

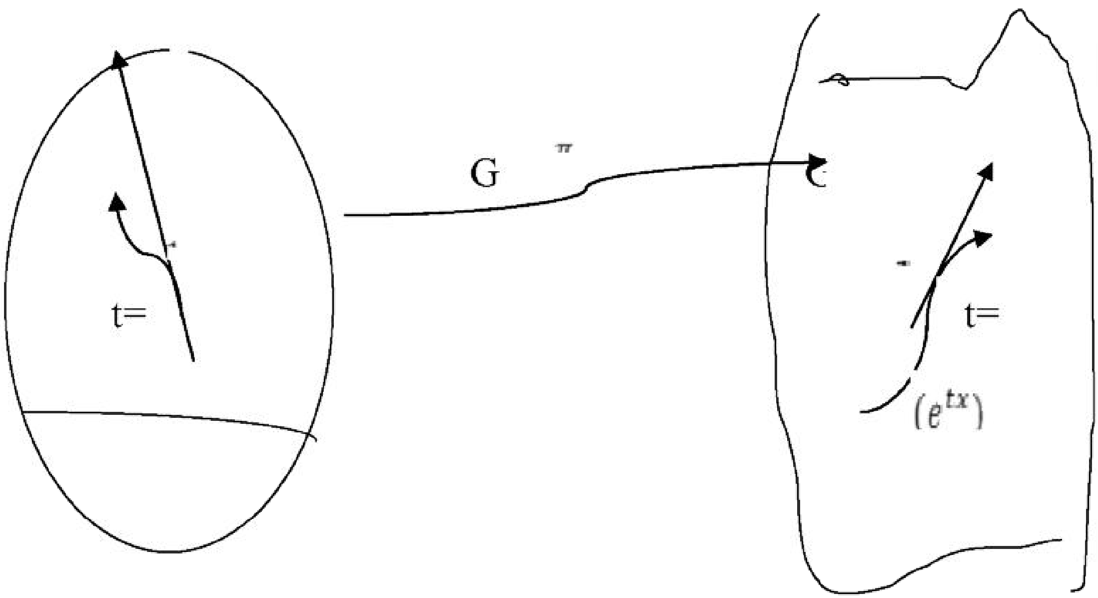
$n$ by $n$ complex matrices $\phi(X_j)$ satisfying the commutation relations

$$[\phi(X_j), \phi(X_k)] = \sum_{l=1}^{d} C_{jkl}\ \phi(X_l)$$

The representation is unitary when the matrices are skew-adjoint.

The notion of Lie algebra is motivated by the fact of homomorphism property causes the map π to be largely determined by its behavior infinitesimally near the identity, and thus by the derivative $\pi'$.

To define the derivative of such a map is in terms of velocity vectors of paths.To representation $\pi : G \to GL(n,C)$ a linear map $\pi': \mathcal{G} \to M(n,C)$ Where $\pi'(x) = \frac{d}{dt}\pi(e^{tX})|_{t=0}$




Fig(2): **Representation the derivative of such a map is in terms of velocity vectors of paths.**

In the case of $U(1)$ we classified all irreducible representations (homomorphism $U(1) \to GL(1,C) = C^*$ ) by looking at the derivative of the map at identity.

**Theorem(3-1) :-**

If $\pi: G \rightarrow GL(n,C)$ is group homomorphism , then

$\pi' : X \in \mathcal{G} \rightarrow \pi'(X) = \frac{d}{dt}(\pi(e^{tX}))|_{t=0} \in gL(n,C) = M(n,C)$

**Satisfies :-**

(1) $\pi(e^{tX}) = e^{t\pi'(X)}$

(2) For $\mathcal{G} \in$G, $\pi'(\mathcal{G} X \mathcal{G}^{-1}) = \pi(\mathcal{G})\pi'(X)(\pi(\mathcal{G}))^{-1}$

(3) $\pi'$is a Lie algebra homomorphism

$\pi'([X,Y]) = [\pi'(X), \pi'(Y)]$

**Proof:-**

(1)we have :

We have : $\frac{d}{dt}\pi(e^{tX}) = \frac{d}{ds}\pi(e^{(t+s)X})\Big|_{s=0}$

$= \pi(e^{tX})\frac{d}{ds}\pi(e^{sX})|_{s=0}$

$= \pi(e^{tX})\pi'(X)$

So $f(t) = \pi(e^{tX})$ satisfies the differential equation $\frac{d}{dt}f = f\pi'(X)$ with initial condition $f(0) = 1$. This has the unique solution $f(t) = e^{t\pi'(X)}$.

(2) We have $e^{t\pi'(\mathcal{G}X\mathcal{G}^{-1})} = \pi\left(e^{t\mathcal{G}X\mathcal{G}^{-1}}\right)$

$= \pi(\mathcal{G}e^{tX}\mathcal{G}^{-1})$

$= \pi(\mathcal{G})e^{t\pi'(X)}\pi(\mathcal{G})^{-1}$

Differentiating with respect to t at t = o gives

$\pi'^{(\mathcal{G}X\mathcal{G}^{-1})} = \pi(\mathcal{G})\pi'(X)(\pi(\mathcal{G}))^{-1}$

(3) Recall that

$$[X,Y] = \frac{d}{dt}(e^{tX}Ye^{-tX})\Big|_{t=0}$$

So $\pi'[X,Y] = \pi'(\frac{d}{dt}(e^{tX}Ye^{-tX})\Big|_{t=0})$

$$= \frac{d}{dt}\pi'(e^{tX}Ye^{-tX})\Big|_{t=0} \text{ (by linearity)}$$

$$= \frac{d}{dt}\pi(e^{tX}\pi'(Y)\pi(e^{-tX})\Big|_{t=0} \text{ (by 2)}$$

$$= \frac{d}{dt}\left(e^{t\pi'(X)}\pi'(Y)(e^{-t\pi'(X)}\right)\Big|_{t=0} \text{(by 1)}$$

$$= [\pi'(X), \pi'(Y)]$$

**[4] The Rotation and Spin Groups in 3 and 4 Dimensions:-**

The rotation group in two dimensions about the origin are given by elements of $SO(2)$, with counter- clockwise rotation by an angle θgiven by the matrix $R(\theta) = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$ .

This can be written as, $R(\theta): e^{\theta L} = cos\theta + Lsin\theta$ for $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

In three dimension the group SO(3) is 3- dimensional and non-commutative . One now has three independent directions one can rotate about, which one can take to be the $X, Y \; and \; Z - axes$ with rotation about these axes given by:

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The infinitesimal picture near the identity of the group, given by the Lie algebra structure on $SO(3)$is much easier to understand.

For orthogonal groups the Lie algebra can be identified with space of anti- symmetric matrices so in this case has basis

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Which satisfy the commutation relations

$[L_1, L_2]= L_3$, $[L_2, L_3]= L_1$, $[L_3, L_1]= L_2$ The Lie bracket operation $(X, Y) \in R^3 \times R^3 \rightarrow [X , Y] \in R^3$

that makes $R^3$ a lie algebra SO (3)= SU(2) is just cross- product on vectors in $R^3$ .

The commutation relations for $L_i$ determine the Lie algebra representation $(ad, SO(n))$ by the definition of the adjoint representation, $(ad(X))_Y = [X, Y]$. For infinitesimal rotations about $X - axis$, one has the adjoint representation

$$(ad(L_1)(L_1) = 0 \,,\ (ad(L_1)(L_1) = L_3 \ , (ad(L_1)(L_1) = -L_2 \quad (2.33)$$

On vectors, such infinitesimal rotations vector, the standard basis $e_i$, of $R^n$ by matrix multiplication, giving

$$L_1 e_1 = 0 \,, L_1 e_2 = e_3, L_1 e_3 = -e_2$$

Lie algebra representation, with the isomorphism identifying $L_i = e_i$. At level of the group, rotations about the x- axis by an angle θ correspond to matrices

$$e^{\theta L_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{pmatrix}$$

Our two isomorphism are on column vectors ( " vector" representation on$R^3$)and on anti- symmetry real matrices ( " adjoint" representation on SO(3) ) with the isomorphism given by :

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \leftrightarrow \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}, \text{let } A = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

On the anti symmetric matrices the Lie group representation is given by $Ad\ (g)\ \ (A)\ \ = \ g\ A\ g - 1$ when g is 3 by 3 orthogonal matrix .

On the anti symmetric matrices the Lie algebra representation given by $ad(X)(A) \ = \ [X\ , A]$ where $X$ is 3 by 3 anti symmetric matrix.

**4-1 Spin Group in Three and Four Dimensions:**

The orthogonal groups $SO(n)$ in that they come with associated group, called spin(n). with every element of $SO(n)$ corresponding to two distinct elements of spin(n) .

$Spin\ (n)$ is topologically the simply- connected double-cover of $SO(n)$ , and one can choose the covering map ϕ: $Spin\ (n)\ \rightarrow\ SO(n)$ to be group homomorphism.

$Spin(n)$ is a Lie group of the same dimension , with an isomorphic tangent space at the identity , so Lie algebras of the two groups are isomorphic $SO(n) \cong spin\ (n)$.

We will construct spin (n) and covering map ϕ only for the cases $n\ = 3$ and n = 4 , with higher dimensional . For n = 3 it turn out that $spin(n)\ =\ SU(2)\ , and\ for\ n\ =\ 4\ , spin\ (4)\ \ =\ SU(2) \times SU(2)$ .

To see how this works it is best to not use just real and complex numbers, but also bring in third number system, quaternions.

**4-2 Rotation and Spin Group in Four Dimensions:-**

Pairs $(u, v)$ are units quaternions give the product group $sp(1) \times sp(1)$. An element of this group on $H = R^4$ by

$q \to uqv$ and this preserves length of vector and is linear in $q$, so it must correspond to element of group $SO(4)$.

The Pairs $(u,v)$ and $(-u,-v)$ give the same orthogonal transformation of $R^4$, so the same element of $SO(4)$. One can show that $SO(4)$ is group $sp(1) \times sp(1)$, with elements $(u,v)$ and $(-u,-v)$ identified. The name $spin(4)$ is given to the Lie group that "double covers" $SO(4)$

So here $$spin(4) = sp(1) \times sp(1)$$

**4-3 Rotations and Spin Groups in Three Dimensions:**

The subgroup spin (3) that only acts on 3 of dimensions and double –covers SO (3).To find this, consider the subgroup spin(4) consisting of pair $(u,v)$ of the form$(u,u^{-1})$ (subgroup isomorphic to $sp(1)$, since elements correspond to a single unit length quaternion u).The subgroup acts on quaternions by conjugation

$$q \to uqu^{-1}$$

So $$q = \vec{v} = v_1 i + v_1 j + v_1 k$$

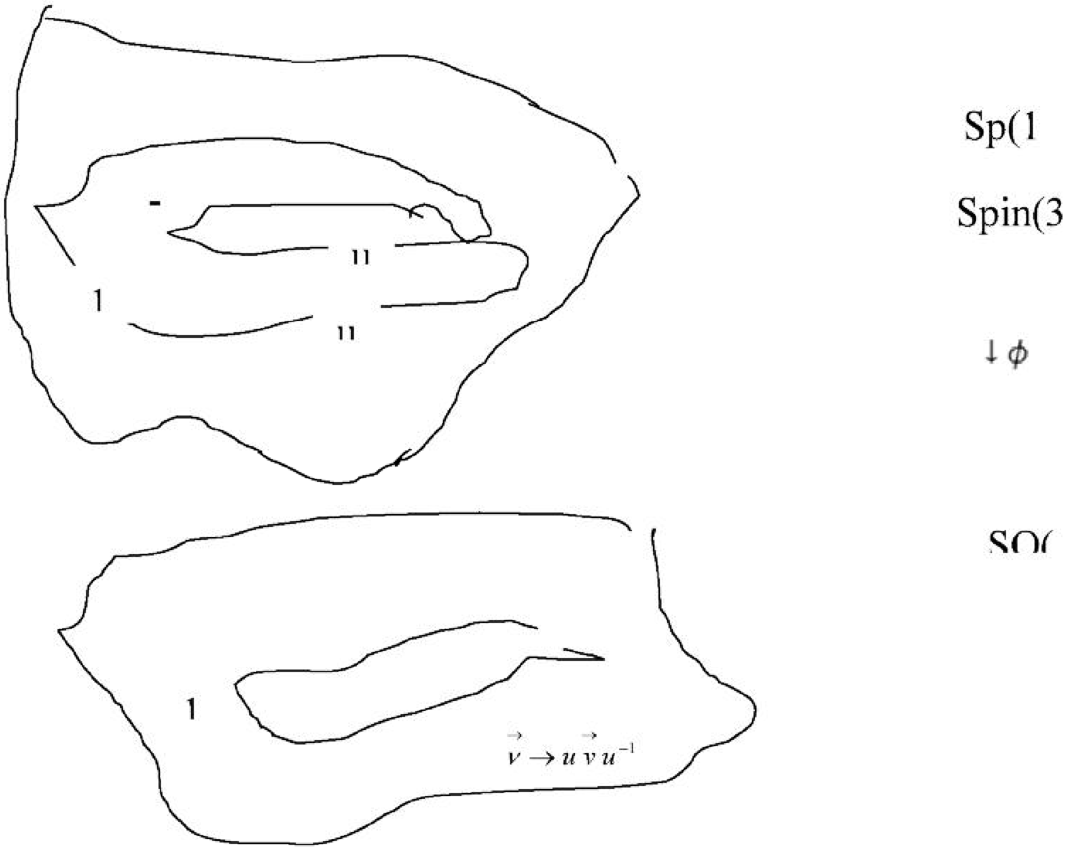
An element $u \in Sp(1)$ acts on $\vec{v} \in R^3 \subset H\ as$

$$\vec{v} \to u\vec{v}u^{-1}$$

This is a linear action, preserving the length $|\vec{v}|$, so corresponds to an element of $SO(3)$ .

We thus have a map

$$\phi\ : u \in sp(1) \to \{\vec{v} \to u\vec{v}u\} \in SO(3)$$




Fig(3):**Mapping between the spin (3) and double –covers SO (3)**.

Both $u$ *and* $-u$ act in same way on $\vec{v}$

The relationship between rotations of $R^3$and unit quaternions is quite simple: for $\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$ a unit vector in $R^3 \subset H$, conjugation by the unit quaternion $u(\theta, \vec{\omega}) = cos\theta + \vec{\omega} sin\theta$

Gives a rotation about the $\vec{\omega}$ axes by an angle 2θ . The factor of 2 here reflects the fact that unit quaternions double –cover the rotation group $SO\ (3)$. For example takes the rotation as Z – axis by choosing $\vec{\omega}$=K. A unit quaternions $u_\theta = cos\theta + ksin\theta$ has inverse

$$u_\theta^{-1} = cos\theta - ksin\theta$$

And acts on $\vec{v} = v_1 i + v_2 j + v_3 k$

$$\vec{v} \to u_\theta \vec{v} u_\theta^{-1} = (cos\theta + ksin\theta)(v_1 i + v_2 j + v_3 k)(cos\theta - ksin\theta)$$
$$= (v_1(cos^2\theta - sin^2\theta) - v_2(2sin\theta cos\theta))i$$
$$+ (2v_1 sin\theta cos\theta + v_2(cos^2\theta - sin^2\theta))j + v_3 k$$
$$= (2v_1 cos2\theta - v_2 sin2\theta)i + (v_1 cos2\theta - v_2 sin2\theta)j + v_3 k$$

For rotations about the Z arises the double – covering map

$$\Phi: u_\theta = (cos\theta + ksin\theta) \in sp(1) = spin(3) \to \begin{pmatrix} cos2\theta & -sin2\theta & 0 \\ sin2\theta & cos2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

As θ goes from 0 to 2π , $u_\theta$ traces a circle in $sp(1)$.

In case of $U(1)$, the unit vector in $R^2$with basis $i$, for the case of $sp(1)$, can again take the identity to be in the real direction, and the tangent space(the Lie algebra $sp(1)$)is isomorphic to $R^3$, with basis $i, j, k$.

The Lie brackets just the commentator e.g. $[i, j] = ij - ji = 2k$ .

Linear combination of these basis vectors one gets for paths

$$u(\theta, \vec{\omega}) = cos\theta + \vec{\omega} sin\theta$$

$$\frac{d}{d\theta} u(\theta, \vec{\omega})|_{\theta=0} = (cos\theta + \vec{\omega} sin\theta)|_{\theta=0} = \vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k$$

The derivative of the map $\phi$ will be linear map $\phi'$:$sp(1) \to SO(3)$ using the formula

$$\phi\,(cos\theta + ksin\theta) = \begin{pmatrix} cos2\theta & -sin2\theta & 0 \\ sin2\theta & cos2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\phi'\,(\mathrm{k}) = \frac{d}{d\theta} \Phi(cos\theta + ksin\theta)|_{\theta=0} = \begin{pmatrix} cos2\theta & -sin2\theta & 0 \\ sin2\theta & cos2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 2L_2$$

Repeating this on other basis vectors one find that $\phi'(i) = 2L_1,\ \phi'\ (j) = 2L_2\ ,\ \phi'\ (k) = 2L_3$ . Thus $\phi'$is an isomorphism of $sp(1)$ *and* $SO\ (3)$ identity the basis $\frac{i}{2}, \frac{j}{2}, \frac{k}{2}$ *and* $L_1, L_2, L_3$

that satisfy simple commutation relations

$$\left[\frac{i}{2}, \frac{j}{2}\right] = \frac{k}{2}, \left[\frac{j}{2}, \frac{k}{2}\right] = \frac{i}{2}, \left[\frac{k}{2}, \frac{i}{2}\right] = \frac{j}{2} \qquad (2.44)$$

**4-4 The Spin Group and SU(2) :**

We discuss the isomorphism between quaternions H and space of 2 by 2 complex matrices, the Pauli matrices can be used to gives such an isomorphism taking

$$I \to 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, i \to -i\sigma_1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$j \to -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, k \to \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix},$$

The correspondence between H and 2 by 2 complex matrices is then given by:

$$\vec{q} = q_0 + q_1 i + q_2 j + q_3 k \leftrightarrow \begin{pmatrix} q_0 - iq_3 & -q_2 - iq_1 \\ q_2 - iq_1 & q_0 + iq_3 \end{pmatrix}$$

Since $$det \begin{pmatrix} q_0 - iq_3 & -q_2 - iq_1 \\ q_2 - iq_1 & q_0 + iq_3 \end{pmatrix} = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

We see that the length squared function on quatemions corresponds to the determinant function on 2 by 2 complex matrices.

The complex matrices in SU (2) can be written in the form $\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ with α, β ∈ C arbitrary complex number satisfying $|\alpha|^2 + |\beta|^2 = 1$ with unit vectors in H is given by $\alpha = q_0 - iq_3, \beta = -q_2 - iq_1$

We see that sp(1) , spin(3) and SU(2) are all names for the same group, geometrically $S^3$ , the unit sphere in $R^4$ .

We have an identification of Lie algebra sp(1) = SU(2) between pure imaginary quaternions skew .Hermition trance– zero 2 by 2complex matrices

$$\vec{\omega} = \omega_1 i + \omega_2 j + \omega_3 k \leftrightarrow \begin{pmatrix} -i\omega_3 & -\omega_2 - i\omega_1 \\ \omega_2 - i\omega_1 & i\omega_3 \end{pmatrix} = \omega.\sigma$$

With basis $\frac{i}{2}, \frac{j}{2}, \frac{k}{2}$ gets identified with a basis for Lie algebra SU(2) which written in terms of Pauli matrices is $X_j = -i\frac{\sigma_j}{2}$ satisfying the commutation relations

$$[X_1, X_2] = X_3,\ [X_2, X_3] = X_1\ ,\ [X_3, X_1] = X_2$$

Which are precisely the same commutation relations as for

SO(3) $[L_1, L_2] = L_3, [L_2, L_3] = L_1\ , [L_3, L_1] = L_2$

We have no less than three isomorphic Lie algebras $sp(1) = SU(2) = SO(3)$ which we have adjoint representation.

$$\omega_1 \frac{i}{2} + \omega_2 \frac{j}{2} + \omega_3 \frac{k}{2} \leftrightarrow -\frac{i}{2} \begin{pmatrix} \omega_3 & -\omega_1 - i\omega_2 \\ \omega_1 + i\omega_2 & \omega_3 \end{pmatrix} \leftrightarrow$$

$$\begin{pmatrix} 0 & \omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

With this isomorphism identifying basis vectors as

$$\frac{i}{2} \leftrightarrow -i\frac{\sigma_1}{2} \leftrightarrow L_1$$

At the level of Lie groups we have seen-that our identification of H and 2 by 2 matrices identifies sp(1) with SU(2) taking

$u(\theta, \vec{\omega}) \rightarrow cos\theta I - i(\omega.\sigma)sin\theta =$

$\begin{pmatrix} cos\theta - i\omega_3 sin\theta & (-i\omega_1 - \omega_2)sin\theta \\ (-i\omega_1 + \omega_2)sin\theta & cos\theta + i\omega_3 sin\theta \end{pmatrix}$

The relation to SO(3) relations is that this is an SU(2) element such that if one identifies vectors $(v_1, v_2, v_3) \in R^3$ with complex matrices $\begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix}$

Then:

$(cos\theta I - i(\omega.\sigma)sin\theta) \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} (cos\theta I - i(\omega.\sigma)sin\theta)^{-1}$

Is the same vector, rotated by an angle 2θ about the axis given by $\omega$.

We will define

$$R(\theta, \omega) = e^{\theta(\omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3)} = e^{-i\frac{\theta}{2}\omega.\sigma} = \cos(\frac{\theta}{2})I - i(\omega.\sigma)\sin(\frac{\theta}{2}) \quad (2.51)$$

And then it is conjugation by $R(\theta, \omega)$ that rotates vectors by an angle θ about $\omega$.

In particular, rotation about the Z-axis by an angle θ is given by conjugation $R(\theta, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}) = \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix}$

In term of the group SU(2) , the double covering map $\Phi$ thus acts on diagonalized matrices as

$$\Phi:\begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \in SU(2) \rightarrow \begin{pmatrix} cos2\theta & -sin2\theta & 0 \\ sin2\theta & cos2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

**4-5 Spin Group in Higher Dimension:-**

For each n>2, the orthogonal group SO(n) is double-covered by group spin(n) with an isomorphic Lie algebra. Special phenomena relating these spin groups occur for n<7 (it turns out that spin(5)=sp(2), the 2 by 2 norm-preserving quaternionic matrices), and spin(6)=SU(4), but in higher dimensions these groups have no relation to quaternions or unitary groups. The construction of the double-covering map $spin(n) \rightarrow SO(n)$.

**4-6 The spinor representation:-**

The irreducible representation is known as spinor or spin representation of spin(3) the homomorphism $\pi_{spinor}$ defining the representation is just the identity map from $SU(2)$ to itself.

The spin representation of spin(3) is not representation of $SO(3)$. The double cover map $\phi : spin(3) \rightarrow SO(3)$ is homomorphism, so given a representation $(\pi, V)$ of so(3) one gets representation $(\pi \circ \phi, V)$ of spin(3) by composition. But there no homomorphism $SO(3) \rightarrow SU(2)$ that would allow us to make the standard representation of $SU(2)$ on $C^2$ into an $SO(3)$ representation. We could try and define a representation of $SO(3)$ by

$$\pi: g \in SO(3) \rightarrow \pi(g) = \pi_{spinor}(\tilde{g}) \in SU(2)$$

Where $\tilde{g} \in SU(2)$ satisfying $\phi(\tilde{g}) = g$

## [5] Conclusions:

The representation theory play an especially important part to explain how rotations in $R^n$ are induced by the action of a certain group, spin(n) on $R^n$.

For a representation $\pi$ and group elements $g$ that are close to the identity, one can use exponentiation to write $\pi(g) \in GL(n, C)$.

spin representation of spin(3) the homomorphism $\pi_{spinor}$ defining the representation is just the identity map from $SU(2)$ to itself.

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