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- Mixed spinor-tensor of rank 2, $Z = Z^{A\dot{A}} O_A \otimes \bar{O}_{\dot{A}}$.

Spinor –tensors associated to Hermitian matrices are called Hermitian also . They span the real Minkowski vector space M. This is vector, or $(½, ½)$ representation. for decomposable spinor tensors , the scalar product is defined through symplectic form, as

$$\eta\left(\xi\otimes\overline{\xi},\ \xi'\otimes\overline{\xi'}\right) = \in (\xi, \xi') \in (\xi, \xi') \qquad (6.2)$$

decomposable spinor-tensors $Z^{A\dot{A}} = \xi^A \bar{\xi}^{\dot{A}}$ corresponds to a null vector (of zero norm) in M_C. To any Weyl spinor ξ is associated the null vector $\xi\otimes\overline{\xi}$ in real Minkowski space-time called its flagpole.

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The symplectic structure $\in$ being preserved by the anti- isomorphism, also allow to raise or lower the dotted indices:

$$\epsilon:\ O^{\dot{A}} \rightarrow O_{\dot{A}}$$

$$\zeta \rightarrow \epsilon(\zeta,.)$$

$$\zeta^{\dot{A}} \rightarrow \zeta_{\dot{A}} = \zeta^{\dot{B}}\epsilon_{\dot{B}\dot{A}}$$

The simplistic form is also preserved: $\Lambda^{-1} \in \Lambda = \overline{\in} = \overline{\Lambda}^{-1} \in \overline{\Lambda}$

(6)Spinor –Tensor and Minkowski Space;

The general element Z of tensor ,product $O^{A\dot{A}} \equiv O^{A} \otimes O^{\dot{A}}$ is called a mixed spinor –tensor of rank 2 . In a simplistic basis, it expands as $Z = Z^{A\dot{A}} O_A \otimes O_{\dot{A}}$ and so is represented by the complex 2×2 matrix Z$\in$ $\mathrm{Mat}_2(C)$ with components $Z^{A\dot{A}}$ using the Pauli matrices as(complex) basis of $\mathrm{Mat}_2(C)$ it expands in turn as $Z = Z^{\mu}\sigma_{\mu}$, $Z^{\mu} \in C$, identifies withthe (complex) vector $Z \in M_{\mathbb{C}}$ with components with

$$Z = Z^{\mu}\sigma_{\mu} = Z^{A\dot{A}}(\sigma^{\mu})_{A\dot{A}} \qquad (6.1)$$

The element of the form $Z = \xi \otimes \overline{\xi} = \xi^{A}\overline{\xi}^{\dot{A}} O_A \otimes \overline{O}_{\dot{A}}$ are called decomposable[4, Blaine]. In matrix rotations.

$Z = \xi\xi^{T}: Z^{A\dot{A}} = \xi^{A}\xi^{\dot{A}}$ (T matrix (or vector) transportations).

This established a one-to-one correspondence between

- Vectors Z in complex Minkowski vector space M_C, $Z = Z^{\mu}\sigma_{\mu} = Z^{A\dot{A}}(\sigma^{\mu})_{A\dot{A}}$
- Complex 2X2 matrix Z$\in$$\mathrm{Mat}_2(C)$ with components $Z^{A\dot{A}}$.

To the frame O^A is associated the co-frame O_A. An element of O_A expands as $\eta = \eta_A O^A$,

For instance, we have $u^A v_A = -u_A v^A$.

The naturally induced (dual) action of element of the spin group, $\Lambda: \lambda \rightarrow \lambda\Lambda^{-1}; \ \eta_{A\rightarrow}\eta_{\beta}(\Lambda^{-1})^{\beta}\Lambda$

, defines the dual representation , that we note spin*.

The complex conjugation isomorphism representation $\overline{spin}$ of group spin on C^2 is defined as

$$\Lambda = \lambda \rightarrow \bar{\lambda}\eta, \quad \eta \in C^2$$

It preserves also the symplectic form ϵ on C^2. We note $\overline{O^A} \equiv O^{\dot{A}}$ this representation vector space. An element is written with dotted indices as $\eta = \left(\eta^{\dot{A}}\right) = \begin{pmatrix} \eta^{\dot{1}} \\ \eta^{\dot{2}} \end{pmatrix}$ where the index $\dot{A}$ takes the values $\dot{1}$, $\dot{2}$ [5, Georgi].

We call $\overline{spin}$ the group acting in this representation , the $D^{(\frac{1}{2},0)}$, or right representation.

The complex conjugation defines the isomorphism (called anti-isomorphism)

$$O^A \rightarrow O^{\dot{A}}$$

$$\zeta = \zeta^A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \mapsto \bar{\zeta} = \bar{\zeta}^{\dot{A}} = \begin{pmatrix} \bar{\alpha} \\ \bar{\beta} \end{pmatrix}$$

We write $\bar{\zeta}$ with dotted indices it belongs to $O^{\dot{A}}$ [11, Eckhard].

$$\in : O^A \times O^A \longrightarrow \mathbb{C}$$
$$\zeta , \zeta \rightarrow \in (\zeta , \zeta)$$

This gives to Weyl – spinor space O^A asymplectic structure **(** C^2**,∈)**

Thus spin appears as symmetry group of the simplistic space O^A.

A frame of O^A is simplistic iff the simplistic form is represented by matrix

$$\in_{AB} = \in (\mu_A , \mu_\beta) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

This justifies the rotation since, in symplectic basis, the component $\in_{AB}$ = identifies with familiar Levi- Civita symbol. In vector rotation

$$\in (\zeta , \zeta) = \zeta^T \in \zeta = \in_{AB} \zeta^A \zeta^B = \zeta^1 \zeta^2 - \zeta^2 \zeta^1 \qquad (5.5)$$

The anti-symmetric form $\in$ defines an anti-symmetric spin–invariant scalar product, called the symplectic scalar product [1,Peter].

Anti symmetry $\Longrightarrow$ symplectic norm, of any spinor is zero: $\in (\zeta , \zeta) = 0$.

The matrix $\in$ is called Levi- Civita spinor, we will consider $\in$ as the expression of the Minkowski metric.

The dual $O_A = (O^A)^*$ of vector space O^A is the space of 1-forms on it. The symplectie form $\in$ on O^A provides a duality isomorphism between both the spaces

$$\in : \; O^A \mapsto O_A \equiv (O^A)^*$$
$$\zeta \mapsto \zeta^* = \in (\zeta , \cdot)$$
$$\mu_A \mapsto \mu^A = \in (\mu_A , \cdot)$$

correspondence as above leads to identify $\mathbb{M}_{\mathbb{C}}$ with the set $Mat_2(C)$ of complex matrices Z=[$Z^{A\dot{A}}$] .

$$C^4 \ni Z(z^{\mu}) \mapsto Z \equiv \begin{pmatrix} z^{1\dot{1}} & z^{1\dot{2}} \\ z^{2\dot{1}} & z^{2\dot{2}} \end{pmatrix} \equiv \begin{pmatrix} z^{\circ} + z^1 & z^2 + iz^2 \\ z^2 - iz^3 & z^{\circ} - z^1 \end{pmatrix} \qquad (5.4)$$

A according to spinorial or twistorial formalism, even more fundamental is its universal covering , the group spin↑ (1, 3)≡ $SL(2, C) = SP(2, C)$.

In its fundamental representation, SL(2, C) is the subgroup of GL(2,C) has complex determinant = 1. Has complex dimension 3 GL(2. C) has complex dimension 4). Thus spin↑ = SL(2, C) act naturally on the vectors of C^2, which are called Weyl spinors or Chiral spinors. This is the so called $D^{(0,½)}$, or left , or negative helicity representation[3, Peskin].

As a vector of the vector space C^2 , a Weyl spinor expands as $\zeta = \zeta^A O_A$ in basis $(O_A) = (O_1 , O_2)$.

Thus it appears as two. Component column vector $\zeta = \begin{pmatrix} \zeta^1 \\ \zeta^2 \end{pmatrix}$ and, by definition, an element of group spin acts a linearly on it, as 2X2 matrix Λ:

$$Spin: C^2 \longrightarrow C^2$$

$$\Lambda: \zeta \rightarrow \wedge\zeta$$

The set of Weyl spinors, with this group action, is written O^A . A Weyl spinor is written ζ^A .

Since $spin \uparrow = Sp(2, C)$ $it\ may$ also be seen as the group of transformation of GL(2, C) which preserve a symplectic form $\in$of C^2:

$x.x = detX, 2x^0 = TrX.$

In the following , we will distinguish usual (x^{μ}) and spinoral ($X^{A\dot{A}}$) coordinate only by the indices[13, Faria].

An element of Lorentz group acts on the Minkowski vector space M as matrices $L: x \to Lx$. The same action is expressed in Herm(2) through a matrices Λ as

$X \mapsto \Lambda X \Lambda^{\dagger}$, Λ is matrices of the group spin the universal covering of the Lorentz group SO. We have the group homomorphism

φ: spin ↦ SO

$\Lambda \quad \mapsto \; L$

We may choose the matrices Λ such that $\Lambda\sigma_{\nu}\Lambda^{\dagger} = L^{\mu}_{\nu}\,\sigma_{\mu}$

Which implies $L^{\mu}_{v} = \frac{1}{2}Tr(\Lambda\sigma_{v}\Lambda^{\dagger}\sigma_{\mu})$.

Note that Λ and -Λ correspond to the same element of the Lorentz group. Which reflect the fact that spin is 1-2. Universal covering of SO [16, Ruhl].

At the infinitesimal level $\Lambda \sim I \sim \lambda, L^{\mu}_{\nu} \sim \delta^{\mu}_{\nu} + L^{\mu}_{\nu}$ so that $\lambda\sigma_{\nu+}\sigma_{\nu}\lambda^{\dagger} \sim L^{\mu}_{\nu}\sigma_{\mu}$

which implies $\lambda = A L^{\mu}_{\nu}\sigma_{\mu}\sigma_{\nu}$

$$A\left(2L^{0}_{i}\sigma_{i} + 2L^{0}_{j}\sigma_{j}\right)^{\sim L^{i}_{0}\sigma_{i}} .$$

The complex MinKowski space time $\mathbb{M}_{\mathbb{C}}$ is defined by the extending the coordinates to complex numbers, and extending the MinKowski metric to corresponding bilinear form $g(z, z') \equiv \eta_{\mu v}\; z^{\mu} z^{v}$. The same spinoral

$$\varphi = (\varphi^A), \dot{\xi} = (\dot{\xi}_A) \to \varphi\,\dot{\xi} = ((\varphi\,\dot{\xi})^A_{\dot{A}} = \varphi^A \dot{\xi}_A) \qquad (5.1)$$

The last relation is a matrix product .It provided a complex matrix of order 2[6, Lawrence].

In the following section, we will study the space-time algebra which is the Clifford algebra of the Spinors in Minkowski space time $M = R^{1,3}$, from group theoretical considerations. We call that the isotropy group of Minkowski space time is orthogonal group $O^{(1,3)}$, with connected components, the restriction to matrices with determinant 1 lead to the special orthogonal group SO(1,3) with 2 connected components. Finally , the components of SO(1,3), connected to the entity is proper Lorentz group SO↑(1,3).

Their (1-2) universal covering are respectively the group pin(1,3), spin(1,3)and spin↑(1,3). The group O, so and SO↑ act on Minkowski space time through the fundamental representation.

The construction of spinors is based on the group isomorphism

spin↑(1,3) = SL(2, C) = SP(2, C) (5.2)

Note also the group isomorphism SO↑(1,3)=SO(1,3).

There is a one to one correspondence between the real Minkowski M and the set of $Herm(2) \subset Mat_{\,2}(C)$ of Hermitian matrices: to any point $x = (x^\mu)$of M is associated the Hermitian matrix

$$X = x^\mu \sigma_\mu = \begin{pmatrix} X^{1\dot{1}} & X^{1\dot{2}} \\ X^{2\dot{1}} & X^{2\dot{2}} \end{pmatrix} = \begin{pmatrix} x^0 + x^1 & x^2 + ix^3 \\ x^2 - ix^3 & x^0 - x^1 \end{pmatrix} \qquad (5.3)$$

Where the σ_μ are Pauli matrices . The matrices coefficients $X^{A\dot{A}}$ with $A = 1, 2, \dot{A} = \dot{1}, \dot{2}$ are spinorial coordinates . We have

(5) Spinor in Minkowski Space-Time:-

We will introduce the spinors of space-time, and later we will link them with the space-time algebra $CL(1,3)$.

To show how spinors appear from purely algebraic point of view. We remark, in $CL(3)$, the two elements (among others) $e_{\pm} = ½(1 \pm e_3)$ are idempotent, i.e., $e_{\pm}^2 = e_{\pm}$.

The set $CL(3)e_{\pm}$ *and* $e_{\pm}CL(3)$are left and right ideals of $CL(3)$. They are vector spaces of (complex) dimension2, and the identification of I to the complex imaginary i makes each of them identical to C^2. A spinor is precisely an element of two dimensional representation space for the group SL(2,C) which is C^2[3, Peskin].

Let us first consider $CL(3)e_+$. If we choose an arbitrary frame (for instance $\binom{1}{0} = e_+, \binom{0}{1} = e_1e_+$)

$\forall\varphi \in CL(3)e_+ , \varphi = \binom{\varphi 1}{\varphi 2} \in C^2$.

We write $\varphi = (\varphi^A)_{A=1,2}$ *and* $CL(3)_{e_+} = O^A$. Such elements constitute a representation, called $D^{(\frac{1}{2},0)}$ of the special linear group SL(2, C). It corresponds to the so called Weyl spinor .

A similar procedure to $e_+Cl(3)$. choosing a basis (e.g.(1,0)=e_+, $(0,1) = e_+e_1$), we write its vectors with covariant (rather than contravariant)

$e_+CL(3) \equiv O_A = \{\zeta \equiv (\zeta_A) \equiv (\zeta_1, \xi_2)\}$. We have the very important mapping

$$O^A \times O_A \rightarrow CL(3)$$

We have also an isomorphism between $CL^{even}(3) and\ CL(0,2)$[12, Atiyah].

(4) Span Minkowski Space-Time

The Para vectors are the Clifford numbers of the form

$$x = x^o 1 + x^i e_i = x^\mu e_\mu (i = 1,2,3) and\ \mu = 0, 1,2,3) \qquad (4.1)$$

This allows us to see the Minkowski space-time as naturally embedded in Clifford algebra of R^3 ,as the vector space of Para vectors $CL_0(3) \oplus CL_1(3)$.

$$x = (x^\mu) = \left(x^0, x^i\right) \longmapsto \bar{x} = (\bar{x}M) \equiv (x^0, -x^i) \qquad (4.2)$$

Allows us to define a quadratic form for the Para vectors

$$Q(x, y) \equiv ½(\bar{x}y + \bar{y}x) = \eta_{\mu\nu} x^\mu y^\nu \qquad (4.3)$$

Where $\eta\ is$ Minkowski norm.

$CL_0(3) \oplus CL_1(3) \qquad \simeq M \qquad \simeq Herm(2).$

(para vectors) $\simeq$ (Minkowski space time) $\simeq$ Herm(2)

C$= x^o + x^i e_i \qquad \simeq (x^\mu)(x^0, x^i) \qquad \simeq m = x^\mu \sigma_\mu$

$Q(c, c) \qquad = \eta(x, x) \qquad = del\ m$

$x^0 \qquad = x^0 \qquad = ½\, Tr\ m.$

The isomorphism between Minkowski Hermitian matrices the three grade 1 vectors , e_i identify with the three ON basis vectors of $R^3 \subset M$ [1, Peter].

- an "imaginary" part $CL^2(3) \oplus CL^3(3) \equiv span\ (Ie_i, I)$
  thus, any pauli number may be seen as complex para vector[7, Varlamov] .

With the identification above (of I by i), the restriction of the multiplication table (5) to the four para-vectors(1, e_i) identifies with that of four Pauli-matrices $(1,\ \sigma_i)_{i=1,2,3} \equiv (\sigma_\mu)_{\mu=0,1,2,3}$. Thus , the real part $CL^{even}(3)$ is isomorphic (as vector space) to Herm(2). The set of Hermitian complex matrices of order 2.

This isomorphism extends to an algebra isomorphism between the complete algebra $CL(3)$ and algebra of complex matrices of order 2. $M_2(C)$, explicited as :

$$1, e_i, Ie_i, I$$

$$1,\ \sigma_i\ , i\sigma_i\ ,\ i$$

The three grade 1 vectors e_i identify with the three traceless Hermitian matrices σ_i .

The algebra isomorphism $CL^{even} \equiv H$ the algebra of quaternions .

Here $CL^{even}(3)$ is algebra of even elements, scalars and bi-vectors.

The isomorphism is realized through $1 \rightsquigarrow j_{o,} Ie_i \equiv e_2 e_3 \rightsquigarrow j_1, -\mathrm{Ie}_2 = -\mathrm{e}_3\mathrm{e}_1 \rightsquigarrow \mathrm{j}_2, \mathrm{Ie}_3 = \mathrm{e}_1\mathrm{e}_2 \rightsquigarrow * \mathrm{j}_3$.

We may extend the isomorphism with $I \rightsquigarrow i$. With the prescription that i commutes with the four j_μ . This allows us to the CL(3) as the set of complex quaternions , $H \times C$.

$$R^2 \rightarrow CL^1(2)$$
$$(x, y) \rightarrow xe_1 + ye_2$$
$$C \rightarrow CL^1(2)$$
$$x + iy \rightarrow xe_1 + ye_2$$

The right multiplication of such a 1-vector by I gives another 1-vector: $(xe_1 + ye_2)I = xe_2 - ye_1$. We recognize a rotation by $\pi/2$ $in\ R^2$.

$V = R^3$, with an on basis $(e_i)_{i=1,2,3}$ construct $CL(R^3) \equiv CL(3)$, the puali-algebra of space. Its elements are sometimes called the Pauli numbers.

The orientation operator:-

The anti symmetrical products of two vectors gives three bivectors (see the table4). The tri-vector $e_1e_2e_3 \equiv I\ with\ I^2 = -1$. The center of $CL(3)$, i.e the set of elements with commute with all elements, is :

$$\mathrm{CL^o(3) \oplus CL^3(3) = span\ (1, I)} \qquad (3.12)$$

The similar algebraic properties of I and of the complex pure imaginary i.

Span $(1, I) \subset CL(3) and$ C$\equiv span\ (1. i)$we may write any bi vector $e_\mu e_v = e_\mu e_v e_p e_p = Ie_p$.

Where the index p is defined the group $\in_{\mu v p}$=1 . This allows us to rewrite the basis of $CL(3)$ under the from $I, (e_i)(Ie_i), I$.

This divides $CL(3)$ into .

- a "real" part $CL^o(3) \oplus CL^3$(3)$\equiv \{para\ vectors\} \equiv span\ (1,\ \mathrm{e_i})$.

[15, Berdt]. Periodicity theorem allow to explore Clifford algebra beyond dimension 8.They obey the following algebra isomorphisms

$$CL(p+1, q+1) \approx CL(1,1) \otimes CL(p, q).$$

$$CL(p+2, q) \approx CL(2,0) \otimes CL(p, q).$$

$$CL(p, q+2) \approx CL(0,2) \otimes CL(p, q).$$

We will pay special attention to

- The algebra of the plane $CL(R^2) = CL(2)$.
- The space algebra, or Pauli algebra $CL(R^3) = CL(3)$.
- The space-time algebra $CL(R^{1,3}) = CL(1,3)$,the algebra of (Minkowski) space-time.

The Clifford algebra of the plane, $CL(R^2) \equiv CL(2)$ extends the 2-dimensional plane $CL(R^2, g)$, with Euclidean scalar product $g(u, v) = u.v. Let\ us\ an\ ON\ basis\ (e_i.e_j = \delta_{ij})\ for\ R^2$.

Anti-symmetry ⇒ the only bi-vector (up to scalar) is

$$e_1 e_2 = e_1 \wedge e_2 = -e_2\, e_1 \equiv I_{CL(2)} = I \qquad (3.11)$$

The rule above imply I^2 = -1 we may check that $CL(2)$ is closed for multiplication, and admits the basis (1,e_1,e_2 ,I) . The general poly vector expands as A = $A^0 1 + A^1 e_1 + A^2 e_2 + A^3 I$

The Euclidean plane R^2 is naturally embedded (as vector space) in $CL(2)$ as $CL^1(2)$, the set of 1-vector.

We have embedding isomorphism

Given a real vector space V, we note $CL(V)$ the complexified Clifford algebra $C\otimes Cl(V)$ [17,Jean Gallier].

A case of interest for physics is when $V = R^{1,3} = M$, Minkowski vector space, and we study below the space time-algebra $CL(M)$. Its complexifiction $CL(M) = CL(R^{1,3})$ is called the Dirac algebra .

More generally from complex algebra $CL(n)$ it is possible to extract the real Clifford algebra $CL(p,q)$ with $p + q = n$. To do so, we extract $R^{p,q}$ $from$ C^n; as complex space, C admits the basis $e_1,\ldots, e_n$.

We may see C as a real vector space with the basis $e_1,\ldots,e_p$, $ie_{p+1},\ldots,ie_{p+q}$. Choosing n vectors in this list, we construct the real sub vectors space $R^{p,q}$, any element $a\in CL(n)$ may decomposed $as\ a = a_r + ia_c$, a_r , $a_c \in CL(p,q)$.

There are natural representations of $CL(d)$ on a (complex) vector space of dimension 2^k , with $k = [\frac{d}{2}]$. It is elements are called Dirac spinors. Element of $CL(d)$ are represented by matrices of order 2^k , i.e., elements of the algebra $Mat_{2k}(C)$, acting as endomorphism[14, Warner].

This representation is faithful when d is even and non- faithful when d is odd.

The structure of a real Clifford algebra is determined by the dimension of the vector space and the signature of the metric, so that it is written $CL_{p,q}(R)$. It is expressed by its multiplication table. A matrix representation of Clifford algebra is an isomorphic algebra of matrices,(such matrix representation lead to the construction of spinor)

For p-form, it coincides with the usual Hodge duality of forms defined from the metric.

In 4-Dimensions, the Hodge duality transforms bivector into a bivector. Any bivector can be decomposed in a self-dual and an anti- dual part:

$$B = B^{+} + B^{-} \quad , \; * B = B^{+} - B^{-} \qquad (3.8)$$

A frame $(e_i)_{i=1....n}$ for V defines a natural frame for $\wedge V$. To define it, we consider all the finite sets of the form

$I \equiv \{i_1, \dots, i_k\} \subset \{i, \dots, n\}$ with $i_1 < i_2 < \cdots < i_k$.

We define the multi vectors $e_I = e_{i_1} \wedge e_{i_2} \wedge \dots \wedge e_{i_k}$ and $e_\theta = e_0 = 1$.

The multi vectors e_I provide a basis for the vectors space $\wedge V$, and thus for C, with the orthographic "index I going from 1 to 2^n".

A multi vectors is expanded in this basis as

$$A = A^I e_I \equiv A_0 + A^i e_i + A^{ij} e_{\{ij\}} + \cdots + A^{1,2,\dots,n} e_{\{1,2,\dots,n\}} \qquad (3.9)$$

Its components A^I may be seen as coordinates in C. Thus function on C may considered as function of the coordinates.

When the basis (e_i) is on $(e_i . e_j = \eta_{ij} = \pm\delta_{ij})$, it is so for the basis (e_I) of $CL\ (V)$, the scalar product of arbitrary multi vectors expands as

$$A \cdot B = \eta_{IJ} \;\; A^I B^J \equiv A^{\circ} B^{\circ} \pm A^i B^i \pm A^{ij} B^{ij} \pm \dots\dots \pm A^{1,2,\dots n} \;\; B^{1,2,\dots\dots n} \qquad (3.10)$$

Summation is assumed over all orthographic indices, and the $\pm$ signs depend on the signature .

When the vector space is complex vector space, its Clifford algebra also complex .

Up to multiplicative scalar, there is a unique $d - multi\ vector$. To normalize, we choose an oriented on basis for V, and define

$I = e_1 \dots e_d = e_1 \wedge \dots \wedge e_d$ as the orientation operator .

It verifies $I^2 = (-1)^{\frac{d(d-1)}{2}+s}$ depending on the dimension and on the signature of the vector space (V, g). The multiples of I are called the pseudo scalars center of $CL(V)$is $CL^0(V)$for d even , or $CL^0(V) \oplus CL^d(V)$for d is odd.

The conjugation, anti automorphsim, is the composition of both: $\bar{R} = (R^*)^T$.

The scalar product of V is extended to $CL(V)$ as

$$g(A, B) = A - B = \langle A^T B \rangle_0 \qquad (3.6)$$

Where $< \cdot >_0$ denotes the scalar part . It is bilinear. It reduces to zero for homogeneous multi vectors of different grades. It reduces to the usual product for scalars (grade 0), to the matrix product for 1- vectors (grade 1).In general

$$A.B = < A >_0 . < B >_0 + < A >_1 .$$
$$< B >_1 + \cdots . + < A >_n . < B >_n \qquad (3.7)$$

The Hodge duality is defined as the operator

$$*: \wedge^p \rightarrow \wedge^{n-p}$$

$$A_p \rightarrow * A_p$$

$$s.t\ B_p \wedge (* A_P) = (B_p . A_p) I , \qquad \forall A_P \in \wedge^p$$

More formally, one may define $CL(V)$ as the quotient of the tensor algebra $T(V)$ over V by the ideal generated by the set $\{x \in V : x \otimes x - g(x,x)1\}$. A poly vector of definite order is called homogeneous, in general, thus is not the case, and we define the projectors $< . >_r$ which project a poly vector onto its homogeneous part of grade r.

We call $CL^k(V)$ the vector space of poly vectors of grade k. As a vector space we have:

$$CL(V) = \bigotimes_{k=0}^{d} CL^k(V) \qquad (3.3)$$

As vector space, we have $CL^0(V) \equiv R$ which is thus seen as embedded in CL(V), as the mult vectors of grade 0(0- vectors) [17,Jean Gallier] .

The vector space V itself may be seen as embedded in CL(V) , as $CL^1(V)$; its elements are the multi vectors of grade 1 or (1-vectors).

The addition of a scalar plus a grade on vector is called a Para vector. It can be expanded as $A = A^0 + A^i e_i$ where $A^0 =< A >_0$ *and* $A^i e_i =< A >$. The vector space of Para vectors is thus $R \oplus V \equiv CL^0(V) \oplus CL^1(V) \subset CL(V)$ (3.4)

We define also the even and odd subspaces of Clifford algebra C as the direct sum

$$C^{even} = \bigoplus_{K\, even} C^k \; and \;\; C^{odd} = \bigoplus_{K\, odd} C^K \qquad (3.5)$$

Both have dimension 2^{d-1} and C^{even} is a subalgebra of C .

The multi vectors belong to the vector space $\wedge V \equiv \bigotimes_{p=0}^{d} \wedge^p V$ all multi vectors, of dimension 2^d [8,Castro].

The wedge product is easily extended to all multivectors by linearity, associatively, distributivity and anti commutatively for the 1- vectors

$$v\wedge w\wedge (v+w+x) = v\wedge w \wedge w + w\wedge w\wedge w + v\wedge w \wedge x$$

$$= (v\wedge v)\wedge w + v(w\wedge w) + v\wedge w\wedge x$$

$$= v\wedge w\wedge x$$

Which is tri-vector if we assume v, w, x linearly independent.

(3) The Clifford Algebra:

We will assume an inner product in V:

$$g: u, v \rightarrow g(u, v) \equiv u \,.\, v$$

one defines the Clifford(or geometrical) product of two vectors as

$$uv \equiv u.v + u\wedge v \qquad (3.1)$$

In general this appears as the sum of a scalar (Poly vector of grade zero) plus a bi-vector (Poly vector of grade 2).

Scalar product $u\wedge v = \frac{uv-vu}{2}$ (3.2)

The Clifford algebra $CL(V)$ is defined as $\wedge V$, with the Clifford product $v, w \rightarrow vw$. As a vector space (but not as algebra) $CL(V)$ is isomorphic to the exterior algebra $\wedge V$. Thus, its elements are the multi vectors defined over V [17,Jean Gallier].

$$\overset{p}{\otimes} V = V \otimes V \otimes \ \dots \otimes V \qquad (2.1)$$

Has also a vector space structure. It is elements, the tensors of (0, p), are sums of element of the form $v_1 \otimes v_2 \otimes \dots \otimes v_p$.

To such a tensor , we associate its completely anti symmetric part

$$v_1 \wedge v_2 \wedge \dots \wedge v_p = skew[v_1 \otimes v_2 \otimes \dots . \otimes v_p]= \frac{\sum[i_1, i_2, \dots, i_p] v_{i_1} \otimes v_{i_2} \otimes \dots \otimes v_{i_p}}{p!} \qquad (2.2)$$

Thus sum extends over all permutations $(p! = \sum(i_1, i_2, \dots, i_p))$. It called wedge (or external) product [10,Talpaert].

Such an external product is skew (0, p) tensor called a p-multi vector (or p-vector). The sum of two p-multi-vector.

If V_p and V_q are p-vector and q-vector, we have

$$V_p \wedge V_q = (-1)^{pq} V_q \wedge V_p \qquad (2.3)$$

(2-2)The Exterior Algebra of Multi Vectors:-

The bi vectors from the vector space $\wedge^2(M)$, of dimension $\frac{d(d-1)}{2}$. A simple bivector B=$a \wedge b$ can be considered as the oriented triangle with vectors $a\ and\ b$ as sides.

Then <B| B> is-the oriented area of the triangle .

Now we extend the sum to multi vectors of different orders, up to d, like

$$A_0 + A_1 + \cdots + A_d$$

Where A_P is P-vector (the expansion stops at d).

products provide canonical basis $(e_{A_1})\otimes(e_{A_2})\otimes\ldots\otimes(e_{A_s})\otimes(e^{B_1})\otimes(e^{B_2})\otimes\ldots\otimes(e^{Br})$ for tensors.

By the direct sum operation, one defines the vector space of all tensors .

$$TV = \bigoplus_{s=0,r=0}^{\infty} T^{(s,r)}V \qquad (1.2)$$

The sets of all tensors of (s, 0) type, $\forall s$ (called covariant) and all tensors of (0, r) type $\forall r$ (called contravariant). An antisymmetric (contravariant) tensor of type (0; p) will be called a p-vector (multi-vector). An anti-symmetric (covariant) tensor of type (p; 0) defines a p-form (multi form).

A vector space V, the anti symmetric part of tensor product of two vectors is defined as

$$v \wedge w = ½(v \otimes w - w \otimes v) \qquad (1.3)$$

This is anti symmetric tensor of rank (0, 2), also called a bivector[17,Jean Gallier]. The wedge product of two vectors defines bivectors, it generalization will lead to consider new objects called multivectors(= skew contravariant tensors). The wedge product is also defined for the dual V^*. The vectors of V^* are called the 1-form of V, the multivectors of $V^*$are called the multi-forms (=skew covariant tensors) of V. With the wedge product, multivectors from an algebra, the exterior (of multi-vectors) $\wedge V$ of V. Multi-forms form the exterior algebra of multi-forms $\wedge V^*$on V.

(2) The wedge product:-

If V is a vector space of dimension d, the tensor product

Clifford Algebras onto MinKowski Space

المستخلص: هدفت هذه الورقة للبحث في جبر كلفرد لايجاد اطار مفهوم لزمرة اللف المغزلي في الفضاء المينكوسكي (الزمكان). عرفنا زمرة اللف المغزلي في زمرة فرعية معينة في وحدات الجبر ، كلفرد الجبر و ارتبط بالفضاء النوني. وأيضا اذا كانت ٣ ≤ ن فان لزمرة اللف المغزلي النونية تبولوجيا أسهل من زمرة SO(n) وأسهل تربطاً.

Abstract: This paper aims to investigate Clifford algebras to get a understanding frame of spinor groups into MinKowski Space Time . The group spin(n) is called spinor group, is defined as a certain subgroup of units of an algebra, CL_n the Clifford algebra and associated with R^n. Furthermore, for $n \geq 3$ the group spin(n) is topologically simpler than the group SO(n). Indeed, for $n \geq 3$, the group spin(n) is simply connected whereas SO(n) is not simply connected.

Keywords: Tensor Algebra, Clifford Algebra, Minkowssski Space, Hermitian Matrix, Weyl Spinor.

(1) Introduction:-

Multi vectors from the exterior algebra of V. Multiform from the exterior algebra of the dual vector space V^*.The definition of the vector space of tensors of type(s, r)

$$\mathrm{T}^{(s,r)} = \overset{s}{\otimes} V^* \overset{r}{\otimes} V \qquad (1.1)$$

Vectors are (0, 1) tensors; 1-forms are (1,0) tensors. An (s, r) tensors is linear operator or $v^s \otimes (v^*)^r$ [4,Bline]. A basis frame(e_A) for V induces canonically a reciprocal basis(co frame) (e^A) for v^* . Their tensor

CLIFFORD ALGEBRAS ONTO MINKOWSKI SPACE

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