References :

[1]E. Cartan , Sur les domains bones homogenous de Icescape de n variable complexes , Abh . Math . Sem. Hamburg 11 (1935) , 116-162 .

[2] L. K. Hua , Harmonic analysis of functions of several complex variables in the classical domains , Science Press ., Peking (1958) , Amer Math .Soc . Transl ., Math . Monograph 6 (1963) .

[3] D-Bekolle . Analysis on tube domains over light cones : some extensions of recent results . Harm . Anal.

[4] D. Bekolle , A . Bonami , Estimates for the Bergman and Szego projections in two symmetric domains of Colloq . Math . 68 (1995) , 81-100 .

[5] D. Bekolle , A. Bonami , M. Peloso , F. Ricci , Boundedness of Bergman projections on tube domains light cones , Math . Zeit , to appear .

[6] C. Fefferman , The multiplier problem for the ball , Ann . of Math .Surveys 19 (1964) , 1-89.

[7] E.M . Stein , Some problems harmonic analysis suggested by symmetric spaces and semi-simple Lie group , Actes du Congres International des Mathematicians , Nice (1970) 1, 173-189 .

[8] D. Bekolle , A. Bonami , G. Garrigos , Littlewood-Paley estimates for holomorphic functions on tube domains over symmetric cones , in preparation .

[9] D. Bekolle , A. Temgoua Kagou , Reproducing properties and L^p -estimates for Bergman projections in Siegel domains of type II, Studia Math . 115 (1995) , 219-239 .

[10] R. Coifman , R. Rochberg , Representation theorems for holomorphic and harmonic functions in L^p , Asterisque , 77 (1980) , 11-66 .

[11]G. Gindikin , Analysis on homogeneous domains , Russia Math . Surveys 19 (1964) ,1-89 .

[12] J. Faraut , A. Koranyi Analysis on Symmetric Cones , Clarendon Press , Oxford (1994) .

[13] D. Bekolle , A. Temgoua Kagou , Molecular decompositions and interpolation , Integr . Equ. Oper . Theory 31 (1998) , 150-177 .

[28] D. Bekoller , A . Bonami , M. Peloso , F. Ricci , Boundedness of Bergman projections on tube domains over light cones , Math .Z. 237. (2001) , 31-59 .

choosing δ small enough , we conclude easily . Now, if we exchange summations , it is sufficient to prove that , for fixed ω , the integral

$$\sum_j \int_{z \in B_j, d_{\Omega(z,\omega)<1}} \frac{dxdy}{Q(y)^n}$$

is uniformly bounded . But it is bounded by $\int_{z \in B_j, d_{\Omega(z,\omega)<1}} Q(y)^n dxdy$ which is a constant by invariance of the distance and the measure . It is easy to deduce the atomic decomposition from the sampling theorem for values of p for which the projection P_v is bounded . More precisely , we get the following theorem .

range $p \in (0, p_0)$ which reduces to $\frac{p\in(0,2(v-1))}{n-2)}$ in the particular case considered here (notice that $2\left(\frac{v-1}{n-2}\right)$ is the upper bound of the range of p for with P_v^+ is bounded on L_v^p; $P_\circ$ has the same property in the general case) . The method given here may be generalized to all tube domains over homogenous cones . Finally, Theorem (1.22)allows to solve a Cartan B problem which we now describe . To simplify , we assume that $n = 3$. We let π^+denote the half-plane of the complex plane $\mathbb{C}$ .For all $p \in (0, \infty)$it is easy to prove that the restriction f of $F \in A_v^p$ given by $f(z_1, z_2) = F(z_1 - z_2, 0, z_1 + z_2)$ belongs to the Bergman space $A_v^p((\pi^+)^2)$. This last space is the subspace of the space $L^p\left((\pi^+)^2, (y_1, y_2)^{v-2} dV(z_1, z_2)\right)$ consisting of holomorphic functions . (It is called a restriction map since it is really given by a restriction when dealing with the spherical cone instead of the future cone) . Moreover , the restriction map

$$A_v^p(\Omega) \to A_v^p((\pi^+)^2)$$
$$F \mapsto f$$

is continuous. We are interested in the range of p for which this map is onto . It has been proved in [13] that it is the case when $p \in [\,2(v-1), 2v-1)$ Moreover , there exists a linear continuous extension map .

Results :

(i)The operator P_v^+ is bounded on L_v^p if and only if

$$\frac{2(v-1)}{2v-n} < p < \frac{2(v-1)}{n-2} - 1 \ .$$

(ii) If P_v is bounded on L_v^p , then

$$\frac{n-2}{2v} < p < 1 + \frac{2v}{n-2}.$$

(iii) P_v is bounded on L_v^p for

$$\frac{n-2}{2(v-1)} < p < 1 + \frac{2(v-1)}{n-2}.$$

(v) The cone is called the Lorentz cone.

Conclusion :

The conclude , it is sufficient to show that the sum of the second term is bounded by the left hand side multiplied by some independent constant C:

choosing δ small enough , we conclude easily . Now , if we exchange summations , it is sufficient to prove that , for fixed ω , the integral

$$\sum_j \int_{z \in B_j, d_{\Omega(z,\omega)<1}} \frac{dxdy}{Q(y)^n}$$

is uniformly bounded . But it is bounded by $\int_{z \in B_j, d_{\Omega(z,\omega)<1}} Q(y)^n dxdy$ which is a constant by invariance of the distance and the measure . It is easy to deduce the atomic decomposition from the sampling theorem for values of p for which the projection P_v is bounded . More precisely , we get the following theorem .

Theorem(1.22)[3]: Assume that P_v is bounded on L_v^p and let $\{z_j\}_{j\in N}$be a δ −lattice in Ω. Then the following assertions hold .

(i) For every sequence $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$\sum_j |\lambda_j|^p Q(y_j)^v < \infty ,$$

the series $\sum_j \lambda_j B_v (z, z_j) Q^v(y_j)$ is convergent in A_v^p . Moreover , its sum Fsatisfies the inequality

$$\|F\|_{A_v^p}^p \leq C \sum_j |\lambda_j|^p Q(y_j)^v .$$

(ii) For δ small enough , every function $F \in A_v^p$ may be written as

$$F(z) = \sum_j \lambda_j Q^v(y_j) B_v(z, z_j) ,$$

with

$$\sum_j |\lambda_j|^p Q\,(y_j)^v \leq C\|F\|_{A_v^p}^p ,$$

Proof : The sampling theorem allows to define a bounded operator from A_v^p into l_v^p the space of sequences $(\lambda_j)_{j\in\mathbb{N}}$ such that

$$\sum_j |\lambda_j|^p Q(y_j)^v < \infty .$$

Using the conjugate exponent and duality ,we prove that its adjoint maps l_v^p into A_v^p . Since the adjoint of the linear form $F \mapsto F(z_j)$ identifies with scalar product with the Bergman kernel $B_v(\cdot, z_j)$,it prove (ii). Moreover , when δ is small enough , we know from the second part of the sampling theorem that A_v^p identifies with a subspace of l_v^p So (ii) is obtained easily using the Hahn-Banach theorem . This theorem allows to have atomic decomposition for $p \in \left(1 + \frac{n-2.}{2(v-1)}, 1 + \frac{2(v-1)}{n-2}\right)$. It had been proved in [13] with another method in the more general case of all symmetric Siegel domains of type II and for two particular affine – homogeneous , nonsymmetric Siegel domain of type II ,but for a

$$, |f(\zeta)-f(z)|^p \leq C\delta^p \int_{d_\Omega(z,\omega)<\delta} |f(\omega)|^p \frac{dudv}{Q(v)^n}, \tag{19}$$

Let us now recall that , as in the case of the distance on the cone , one can find Whitney decompositions of Ω . More precisely, the following lemma , which is given in [10] , is the exact analogue of Lemma (1.3) (and the same proof givens it).

Lemma (1.20)[3]: There exists a positive integer N such that , given $0<\delta<1$, one can find a sequence of points $\{z_j\}$ in Ω with the property that, if we call B_j and B_j' the Bergman balls with center $z_j$and d_Ω −radius δ and $\frac{\delta}{2}$ respectively , then

(i) the balls B_j' are pairwise disjoint ;

(ii) the balls B_j cover Ω and are almost disjoint in the sense that each point belongs to at most $N$of these balls .Following Coifman and Rochberg , we say that the sequence $\{z_j\}_{J\in\mathbb{N}}$ given in the lemma above is an δ −lattice in Ω . We can now state the sampling theorem for functions in A_v^p.

Proposition (1.21)[3]: Let $\{z_j\}_{J\in\mathbb{N}}$ be a $\delta-$ lattice in Ω, with $z_j = x_j + iy_j$. Then , there exists a constant C_δ such that , for $f\in A_v^p$ one has following inequality

$$\sum_j |f(z_j)|^p Q(y_j)^v \leq C_\delta \|F\|_{A_v^p}^p ; \tag{20}$$

Moreover , if δ is small enough , the converse inequality

$$\|F\|_{A_v^p}^p \leq 2C_\delta \sum_j |f(z_j)|^p Q(y_j)^v$$

is also valid .

Proof . The first inequality follows from the mean value inequality (18) applied to the balls $B_j'$and the fact that , on B_j' , $Q(y)$ is equivalent to $Q(y_j)$. For the second inequality , we write that

$$\int_\Omega |f(z)|^p Q(y)^{v-n} dxdy$$
$$\leq c_p \sum_j Q(y_j)^v \int_{B_j} \left(|f(z_j)|^p + |f(z)-f(z_j)|^p\right) \frac{dxdy}{Q(y)^n} .$$
$$\leq$$
$$c_p \sum_j |f(z_j)|^p Q(y_j)^v +$$
$$c_p\delta^p \sum_j \int_{B_j} \left(\int_{d_{\Omega(z,\omega)<1}} |f(\omega)|^p Q(v)^{v-n} dudv\right) \frac{dxdy}{Q(y)^n}.$$

Tom conclude , it is sufficient to show that the sum of the second term is bounded by the left hand side multiplied by some independent constant C:

α , we use complex interpolation for the analytic family of operators $\tilde{T}_\beta$, using the estimate for $a = 0$, and the estimate given in Proposition (1.11) for a value of β with a large real part . Let us remark that it is important , at this point, that norms increase only exponentially when imaginary part of β tends $\pm\infty$. This given by estimates of Lemma (1.9) and a careful study of the constants. In this section , we recall some applications of the boundedness of the Bergman projection . The first one is a direct one , and deals with duality . More precisely ,

Lemma (1.19)[5]: Let $p, q \in (1, \infty)$. If P_v extends into a bounded operator on $L_v^{p,q}$, then the topological dual $\left(A_v^{p,q}\right)'$of $A_v^{p,q}$ identifies with $A_v^{p',q'}$by means of the map:

$$G \in A_v^{p',q'} \mapsto L_G(F) = \int_\Omega F(z)\overline{G(z)}\, Q^{v-n}(\Im m z) dV(z) .$$

Using Theorem (1.1) , we see that the topological dual of A_v^p identifies with $A_v^{p'}$by means of the map (17) when

$$\frac{n-2}{2(v-1)} < p < \frac{n-2}{2(v-1)}.$$

Before going on , let us remark that the domain Ω may also be identifies to a group . More precisely , for $(u, g) \in \mathbb{R}^n \times H$, we can define an automorphism of Ω by the action $z \mapsto gz + u$. Moreover , the product of two such automorphisms has the same form , which allows to define the product of two elements of $\mathbb{R}^n \times H$ .This last one is see as a semi-direct product. Its action on Ω is simply transitive.The measure $Q(y)^{-n} dxdy$ is invariant under the action $\mathbb{R}^n \times H$. Next, let d_Ω denote the Bergman distance on Ω . It is invariant under the action of $\mathbb{R}^n \times H$ and equivalent to the Euclidean distance in a neighborhood of $i\underline{e}$. From the mean value inequality for holomorphic functions , we get that there exists a constant C such that , for f holomorphic in Ω , $p > 0$, $\delta < 1$ and $d_\Omega\left(i\underline{e}, z\right) < \delta$,

$$\left|f\left(i\underline{e}\right)\right|^p \leq C\delta^{-n} \int_{d_\Omega(i\underline{e},z)<\delta} |f(\omega)|^p \frac{dudv}{Q(v)^n},$$

$$\left|f\left(i\underline{e}\right) - f(z)\right|^p \leq C\delta^p \int_{d_\Omega(i\underline{e},\omega)<\delta} |f(\omega)|^p \frac{dudv}{Q(v)^n},$$

Using the action of $\mathbb{R}^n \times H$ it follows that

$$|f(z)|^p \leq C\delta^{-n} \int_{d_\Omega(z,\omega)<\delta} |f(\omega)|^p \frac{dudv}{Q(v)^n}, \quad (18)$$

and , for $d_\Omega(\zeta, z)$

$$\sum_j \frac{\left(\int_{B_j} |g(\xi)|^2 d\xi\right)^{\frac{p}{2}}}{Q(\xi_i)^{v-\frac{n}{2}+\frac{np}{2}}} < \infty;$$

We refer to [5] for a proof of theorem (1.16) when $p > 1$ which is the case that we use later . A key point , for the proof , is the density of $A_v^2 \cap A_v^{p,2}$ into $A_v^{p,2}$. More generally , we have

Lemma (1.17)[3]: For all $p, q \in [1, \infty)$ the intersection of any two space $A_v^{p,2}$ and $A_\mu^{r,s}$ with $1 \leq p, q, r, s < \infty$, is dense in each of them . A new proof of Theorem (1.16) , as well as new developments , are in progress in a joint work with G. Garrigos . In particular , we prove in [8] that the condition on p is critical (actually it is proved in [5] for p an even integer). From this theorem, it is clear that the operator $\Theta_{\alpha-\beta}$, for positive $\beta > a$, extend to bounded operator from $A_{v+\beta p}^{p,2}$ onto $A_{v+\beta p}^{p,2}$ when $p < \frac{4(v+ap-1)}{n-2}$.Using Lemma (1.14) and the fact that T_β is bounded for β large enough as well as similar properties for the adjoint, we obtain that $\tilde{T}_\beta$ is bounded on $L_v^{p,2}$ whenever

$$\frac{n-2-4a}{4(v-1)} < \frac{1}{p+1} < \frac{2v-n}{2(v-1)}.$$

The necessity of these conditions can be proved using the fact that the condition on p is critical in Theorem (1.16) If we interpolate these results with the L^p estimates coming from Proposition (1.11) , we obtain the next proposition , which generalizes Part (iii) in Theorem (1.1) . To simplify the statements , we restrict to non negative values of a .

Proposition (1.18)[3]: Let $a = \Re e\alpha$ be non negative. Then T_α extends into a bounded operator from $L_v^{p,q}$ into $L_{v+ap}^{p,q}$ if the following inequalities are satisfied

$$\frac{n-2}{2q} - \frac{v-1}{p+1} < a\ ;$$
$$\frac{n-2}{2q} + \frac{v-1}{p+1} > \frac{n-2}{2} - a\ ;$$
$$\frac{n-2}{2q'} - \frac{v-1}{p'} < 0\ ;$$
$$\frac{n-2}{2q'} + \frac{v-1}{(p+1)'} > \frac{n-2}{2}.$$

Proof . Assume that $p > 0$ .Then we use interpolation between $L_v^{p,2}$ estimates given above , and $L_v^{p,1}$ or $L_v^{p,\infty}$ estimates given in Proposition (1.11) . By duality, for $a = 0$, we get estimates when $p < 2$. For general

$$v > \frac{n}{2q} + \frac{v-1}{p}$$
$$v + a > \frac{n}{2q'} + \frac{v-1}{p'} \cdot \frac{n-2}{2q} - \frac{v-1}{p} \leq a\ ;$$
$$\frac{n-2}{2q'} - \frac{v-1}{p'} \leq 0\ .$$

Proof: Let us test the operator on the function $f(x+iy) = Q(y)^{n-v}\chi_{|z-i\underline{e}|<\frac{1}{2}}(z)$. By the mean value property of the antiholomorphic functions ,$T_\alpha f\ (z) = Q\big(z + i\underline{e}\big)^{-v-\alpha}$. We get the first condition when using the necessary and sufficient conditions given in Lemma(1.9) . We get the next one , using the same method for the adjoint operator of $\tilde{T}_\alpha$. For the next conditions , we shall use a family of test functions .For $\delta + v - n > -1$ and $\gamma > \frac{n}{2} - 1$, we consider

$$f_{\delta,\gamma}(\omega) = \frac{Q(v)^\delta}{\left[-Q\big(u+i(v+\underline{e})\big)\right]^\gamma}.$$

Using (10) , we know that the Fourier transform , in the u −variable , of $f_{\delta,\gamma}$ is equal to

$$Q(v)^\delta e^{-\xi\cdot(v+\underline{e})} Q(\xi)^{\gamma-\frac{n}{2}}$$

up to a constant .For the same reasons, the Fourier transform of $Q\big(x + i(y+v)\big)^{-v-n}$ is equal to

$$e^{-\xi\cdot(y+v)} Q(\xi)^{v+\alpha-\frac{n}{2}}$$

Using the fact that T_α acts as a convolution operator in the x −variable , and the identity

$$\int_\Gamma e^{-2\xi\cdot v} Q(v)^{\delta+v-n} dv = cQ(\xi)^{-\delta-v+\frac{n}{2}},$$

we obtain that

$$T_\alpha f_{\delta,\gamma} = c\left(-Q\left(x + i\big(y + \underline{e}\big)\right)\right)^{-\alpha-\gamma+\delta}$$

Then $f_{\delta,\gamma}$ belongs to $L_v^{p,q}$ if and only if $\gamma > \frac{n-1}{q}$,$\delta > -\frac{v-n+1}{p+1} - 1$ and $\gamma - \delta > \frac{n}{2q} + \frac{v-1}{p+1} - 1$, a while $\tilde{T}_\alpha\ f_{\delta,\gamma}$ belongs to $L_v^{p,q}$ if and only if $\gamma + a - \delta > \frac{n-1}{q}$, and $\gamma - \delta > \frac{n}{2q} + \frac{v-1}{p} - 1$. The last one is obtained when using the same test functions and the adjoint operator of $\tilde{T}_\alpha$. Let us now consider sufficient conditions. They are based on the following characterization of the space $A_v^{p,2}$.

Theorem (1.16)[3]: For $-2 < p < \frac{2(2v-3)-n}{n-2}$ a function F belongs to $A_v^{p,2}$ if and only

Laplace transform of $e^{i\bar{z}.\xi}Q^{v+\bar{\alpha}}(\xi)$.So, using the polarization of (13) , we can write $T_\alpha(F)(z)$ as the scalar product , in $L^2_{-v}(\Gamma)$, of $g(\xi)$ and $e^{i\bar{z}.\xi}Q^{v+\bar{\alpha}}(\xi)$. We recognize the definition of the Laplace transform of $Q^\alpha g$, which we wanted to find . The constants have been chosen so that we get the identity for $\alpha = 0$, and P_v identifies with $T = T_0$. Moreover , the fact that the restriction of $T_\alpha$to A_v^2 gives an isometry (up to a constant) follows directly from (16) and (13) .

The following lemma is an easy consequence of the previous identity . We note Θ_α the restriction of T_α to A_v^2 ,which , by (16) , is given by the multiplication by $Q^\alpha(\xi)$ of the spectral function . It extends clearly to a well defined operator on all spaces A_μ^2 . Let us remark that , for α a positive integer , it coincides with powers of the D' Alembert operator . This one had been the only case considered in [5] . We have

Lemma (1.14)[3]: The following identities are valid .

$$\Theta_\alpha \circ P_v = T_\alpha \ , \quad \Theta_{\alpha-\beta} \circ T_\beta = T_\alpha \ .$$

Proof . Let $\tilde{T}_\alpha$ maps L_v^2 into itself , and its adjoint maps clearly L_v^2 into A_v^2 since its kernel is holomorphic. It follows that $P_v \circ \tilde{T}_\alpha^* = \tilde{T}_\alpha^*$, which gives the first identity. The second one follows from the first one , using the fact that$P_v = \Theta_{-\beta} \circ T_\beta = \Theta_{-\alpha} \circ T_\alpha$. At this point , let us remark that we have chosen to fix parameter v once for all , and defined the analytic family T_α in terms of the weighted measure $Q(y)^{v-n}dxdy$. Up to multiplication by $Q(y)^{ib}$, the adjoint operator of $\tilde{T}_\alpha$ belongs to the analytic family related to the weighted measure $Q(y)^{v-n}dxdy$ (and has parameter $\bar{\alpha} - a$ in this new family). So, from L^p estimates for the analytic families T_α related to all parameters v , we get also weighted L_v^p inequalities for the analytic family T_α related to a fixed parameter v. We have already found the best estimates for the operators T_a^+ in the last section . When there is an estimate for T_a^+ , the same estimate is valid for the operators T_α (with α of real part a). We are now interested in estimates for T_α which do not extend to the positive operator T_a^+ . Let us first give necessary condition (ii) in Theorem (1.1) .To simplify the statements , we restrict to non negative values of a .

Proposition (1.15)[3]: Let $a = \Re e\alpha$ be non negative. Assume that T_α extends into a bounded operator from $L_v^{p,q}$and $L_{v+ap}^{p,q}$.Then , the following conditions are satisfied

Let us go back to the general Laplace transforms

$$\mathcal{L}\, g(z) = \int_\Gamma e^{iz.\xi}\, g(\xi)\, \frac{d\xi}{Q(\xi)^{\frac{n}{2}}}$$

Assume that g is in $L^2_{-v}(\Gamma) = L^2\left(\Gamma, Q(y)^{-v-\frac{n}{2}}dy\right)$.Using first the Plancherel formula for the L^2 norm in the x variable , then (10) , it is easy to compute the norm of $\mathcal{L}\, g$ in in fact the converse is also valid , and one has the following Paley-Wiener type theorem([12]or[13]):

Proposition(1.12)[15]: A function F on Ω is in A^2_v if and only if $F = \mathcal{L}\, g$, with $g \in L^2_{-v}(\Gamma)$. Moreover ,

$$\|F\|^2_{A^2_v} = 2^{-\frac{n}{2}}(2\pi)^{\frac{3n}{2}-1}\Gamma\left(v-\frac{n}{2}\right)\Gamma(v-n+1)\|g\|^2_{L^2_{-v}(\Gamma)} \ . \qquad (13)$$

It is easy to deduce the Bergman kernel from such a proposition . We claim that

$$B_v(z,\omega) = c_v(-Q)^{-v}(z-\overline{\omega}) , \qquad (14)$$

with

$$c_v = 2^{2v}(2\pi)^{-2n}\Gamma(v)\Gamma\left(v+1-\frac{n}{2}\right)\left[\Gamma\left(v-\tfrac{n}{2}\right)\Gamma(v-n+1)\right]^{-1}.$$

To prove it , let us consider the operator $Tf(z) = c_v\int_\Omega(-Q)^{-v}(z-\overline{\omega})f(\omega)Q(v)^{v-n}dudv$. We use the notations $z = x+iy$ and $\omega = u+iv$. We prove now that T coincides with P_v . Indeed , we know from Proposition (1.11)that $T$is a bounded operator L^2_v . Moreover it is clearly self-adjoint . So it is sufficient to prove that it coincides with the identity on A^2_v .We shall prove a more general statement, which applies to a whole analytic family of operators . For $\alpha = a+ib$, with $a > -\frac{v-n+1}{2}$,we consider the operator

$$T_\alpha f(z) = c_v\int_\Omega(-Q)^{-v-\alpha}(z-\overline{\omega})f(\omega)Q(v)^{v-n}dudv \ . \qquad (15)$$

From proposition (1.11) we know that $T_\alpha$is a bounded operator from L^2_v to L^2_{v+2a} . We claim the following .

Proposition (1.13)[3]: For $a > -\frac{v-n+1}{2}$,the operator T_α defines , up to a constant , an isometry of A^2_v ontoA^2_{v+2a} . Moreover , for $F = \mathcal{L}\, g \in A^2_v$, the function $T_\alpha(F)$ may be written as

$$T_\alpha(F) = \beta_\alpha\mathcal{L}\,(Q^\alpha g) \ . \qquad (16)$$

Proof . Let us show (16) . For $F = \mathcal{L}\, g \in A^2_v$ and $z \in \mathbb{C}^n$, we can see $T_\alpha F(z)$ as the scalar product of F with the function $(-Q)^{-v-\overline{\alpha}}(.-\overline{z})$ in A^2_v .We know from (10) that the second one is the

Let remark that , as a consequence , we get part (i) of Theorem(1.1) for $a = 0$.
The lower bound for p does not depend on a , while the upper bound is ∞ when $a > \frac{n}{2} - 1$.

Proof : Let us first show the sufficient condition . Clearly , T_a^+ acts as convolution operator in the x, u variable . Moreover , the norm of this convolution operator acting in $L^q(\mathbb{R}^n)$ is bounded by the L^1 norm of the kernel$\left|Q\left(x + i(y + v)\right)\right|^{-v-a}$in the x variable , which by Lemma (1.9) is bounded by $cQ(y + v)^{-v-a+\frac{n}{2}}$ .Using also Minkowski inequality , we see that

$$\left(\int_{\mathbb{R}^n} |T_a^+ f(x + iy)|^q dx\right)^{\frac{1}{q}} \leq c \int_\Gamma Q(y + v)^{-v-a+\frac{n}{2}} dv,$$

By assumption , F belongs to $L^p_{v-\frac{n}{2}}(\Gamma)$, and has norm equal to the norm of f in $L_v^{p,q}$. To conclude , we use the sufficient condition in Lemma (2.1.6) to prove that the operator of kernel $Q(y + v)^{-v-a+\frac{n}{2}}$ maps $L^p_{v-\frac{n}{2}}(\Gamma)$ into $L^p_{v-\frac{n}{2}+ap}(\Gamma)$ under the assumptions on a, v, p. Let us now show the necessary condition. We test the operator T_a^+on functions $f(x + iy) = \chi_{|x|<2}(x)g(y)$, with g a positive function supported in the intersection of the cone with the Euclidean ball of radius $1/4$ centered at 0 . Using Lemma (1.10), it follows that for x, y such that $|x|, |y| < 1/4$, one has the following inequality

$$T_a^+(x + y) \geq c \int_\Gamma Q(y + v)^{-v-a+\frac{n}{2}} g(v) Q(v)^{v-n} dv, .$$

By assumption , there exists a constant C independent of g, such that

$$\int_{y\in\Gamma', |y|<\frac{1}{4}} \left(\int_\Gamma Q(y + v)^{-v-a+\frac{n}{2}} g(v) Q(v)^{v-n} dv\right)^p (y + v)^{v+ap-n}$$
$$\leq C \int_\Gamma g(v)^p Q(v)^{v-n} dv.$$

By homogeneity of the kernel, we can replace the constant $1/4$ by any positive constants : for every positive function on , we have the inequality

$$\int_{y\in\Gamma, |y|<N} \left(\int_\Gamma Q(y + v)^{-v-a+\frac{n}{2}} g(v) Q(v)^{v-n} dv\right)^p Q(y)^{v+ap-n} dy$$
$$\leq C \int_{v\in\Gamma, |v|<N} g(v)^p Q(v)^{v-n} dv.$$

Using the density of compactly supported functions, we get the same inequality without any bound on integrals . The necessary condition of the proposition is then a consequence of the necessary condition in Lemma (1.6).

$$\alpha > max\left\{\frac{n-1}{q}, \frac{n}{2q} + \frac{v-1}{p}\right\}.$$

Moreover ,

$$\|F_\alpha\|_{A_v^{p,q}} \leq C_a exp(\pi|b|)Q(t)^{-a+\frac{n}{2q}+\frac{n}{2p}}$$

Proof : Write $J_\alpha(y)$ as the norm in the x variable of the function$(-Q)^{-\frac{\alpha}{2}}(x+iy)$. By (10) and Plancherel's formula , it is finite if and only if $e^{-y.\xi}Q(\xi)^{\frac{(a-n)}{2}}$is in the space $L^2(\Gamma)$ .This shows (i) , using Lemma (1.5). Property (ii) follows also from Lemma (1.5),using the fact that , since the principal argument of $-Q$ is contained is contained in the interval $(-\pi, +\pi)$, we have the inequality $|(-Q)^{-\alpha}(z+it)| \leq |(-Q)^{-\alpha}(z+it)|exp(\pi|b|)$. In fact , one can write a more precise bound below for integrals of $|Q(x+iy)|^{-\alpha}$. It will be used in the proof necessity for P_v^+ in Theorem (1.1) .

Lemma (1.10)[3]: For $\alpha > n-1$, there exists a constant C such that , for every $y \in \Gamma$ with $|y| < 1/2$, one has

$$\int_{|x|<1} |Q(x+iy)|^{-a} dx \geq CQ(y)^{-a+\frac{n}{2}}.$$

Proof : The proof given here is different from the proof given in [4] . We know from Lemma (1.4) that it is sufficient to prove the inequality

$$\int_B |Q(x+iy)|^{-a} dx \geq CQ(y)^{-a+\frac{n}{2}}.$$

where B is the ball of d −radius δ which is centered in y. Now, we can use the fact that Q is almost constant on this ball , which allows to write that the left hand side is equivalent to

$$Q(y)^{\frac{n}{2}} \int_B |Q(x+iy)|^{-a} \frac{dx}{Q(y)^{\frac{n}{2}}}.$$

Using the action of H and the formula of change of variable for Q, we see that this last quantity is equal $Q(y)^{-a+\frac{n}{2}}$ multiplied by the same integral when computed for $y = \underline{e}$ .This last factor is clearly a positive . Let us consider the operator T_a^+ defined by

$$T_a^+ f(z) = \int_\Omega |Q(z-\overline{\omega})|^{-v-a} f(\omega)Q(v)^{v-n} dudv, \qquad (12)$$

where we have used the notation $\omega = u + it$ .The next proposition gives the necessary and sufficient so that T_a^+ is bounded from $L_v^{p,q}$ to $L_{v+pa}^{p,q}$.

Proposition (1.11)[3]: The operator $L_v^{p,q}$is a bounded operator from $L_v^{p,q}$ to $L_{v+pa}^{p,q}$ if only if the following conditions are satisfied

$$\alpha > \frac{n-1-v}{p}, \quad max\left\{-\frac{a}{v-n+1}, \frac{n-2-2a}{2(v-1)}\right\} < \frac{1}{p} < \frac{2v-n}{2(v-1)}.$$

Here, for $\mathcal{Z} \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^n$, we note $z\,.\,\xi = z_1\xi_1 + \cdots + z_n\xi_n$.
The Laplace transform is well defined and holomorphic in the tube domain Ω under integrability conditions on g.Moreover, the function g is uniquely defined knowing $\mathcal{L}\, g$, and is called the spectral function of $\mathcal{L}\, g$.
Let us start with the fundamental example given by powers of Q. More precisely , we still denote by Q.the holomorphic polynomial on $\mathbb{C}^n$given by

$$Q(z) = z_n^2 - z_1^2 - \cdots - z_{n-1}^2$$

The fact that arbitrary powers of Q are well defined is given by the following elementary lemma .

Lemma (1.7)[3]: The image of Ω under the function $-Q(z)$ is contained in $C\backslash(-\infty, 0]$ and is strictly positive on $i\Gamma$, Also , for $y \in \Gamma$, $|Q(x + iy)| \geq Q(y)$. Hence , for $\alpha \in \mathbb{C}$, we shall denote by $(-Q)^{\alpha}(z)$ the determination of the α −th power which corresponds to the determination of the logarithm that is real on $i\Gamma$. Under restrictions on the parameter α , these functions may be written as Laplace transforms of different powers of the function Q restricted to the cone .The following lemma is proved in [12] in a more general setting (one uses the same coordinates as in the proof of Lemma(1.5)).

Lemma (2.8)[15]: The integral

$$\int_{\Gamma} e^{iy.\xi} Q(\xi)^n \frac{d\xi}{Q(\xi)^{\frac{n}{2}}}.$$

converges for $y \in \Gamma$ if and only if $\Re e\alpha > \frac{n}{2}{-}1$. For these values of α and for z, the following identity holds :

$$\int_{\Gamma} e^{iz.\xi} Q(\xi)^{\alpha} \frac{d\xi}{Q(\xi)^{\frac{n}{2}}} = 2^{2\alpha-\frac{n}{2}}\Gamma(\alpha)\Gamma\left(\alpha - \frac{n}{2} + 1\right)(-Q)^{-\alpha}(z). \qquad (10)$$

It means that the holomorphic function$(-Q)^{-\alpha}$ is the Laplace transform of the function Q^{α}on the cone . Then the next lemma gives necessary and sufficient conditions on the power α for the function to belong to the space $A_v^{p,q}$.

Lemma (1.9)[3]: Let $\alpha = a + ib \in \mathbb{C}$. Then

(i) the integral

$$J_{\alpha}(y) = \int_{R^n} |Q(x + iy)|^{-a} dx \qquad (11)$$

converges if and only if $\alpha > n - 1$ and , in this case $J_{\alpha}(y) = C_a Q(y)^{-a+\frac{n}{2}}$;

(ii) for $t \in \Gamma$, the function $F_{\alpha}(z) = (-Q)^{-\alpha}(z + it)$ belong to $A_v^{p,q}$ if and only if

The following lemma given continuity properties of $S_{(\alpha+1),(\beta+1)}$ in $L^{p+1}(\Gamma)=L^{p+1}(\Gamma,dv)$.

Lemma (1.6)[3]: The operator $S_{(\alpha+1),(\beta+1)}$ is a bounded operator in $L^{p+1}(\Gamma)$ if and only if then following conditions hold:

$$-(\alpha+1)<\frac{1}{p+1}<\frac{n+2((\beta+1))}{2n-2},\qquad -(\beta+1)<\frac{1}{p+1}<\frac{n+2(\alpha+1)}{2n-2},$$

Proof : We show the necessary conditions. If we take the characteristic function of the ball of d −radius $(\delta+1)$ centered at $\underline{e}$ as a test function f, we know from Lemma(1.2) that $Q(v)$ and $Q(y+v)$ are almost constant on the support of f. So $S_{(\alpha+1),(\beta+1)}$, is bounded in L^{p+1}, the function $Q(y)^{(\alpha+1)}Q(y+v)^{-(\frac{n}{2}+\alpha+\beta+2)}$ is in $L^{p+1}(y)$. Using the previous lemma , this gives half of the conditions . The two other ones are obtained by the same method , using the fact that the adjoint of $S_{(\alpha+1),(\beta+1)}$ is bounded in $L^{(p+1)'}(y)$,with $(p+1)'$ the conjugate exponent . To show the sufficient conditions , we use Schurs lemma , and test on functions $\phi(v)=L(v)^{\gamma+1}Q(v)^{(\gamma+1)'}$. More precisely , it is sufficient to show that it is possible to choose $(\gamma+1)$and $(\gamma+1)'$ so that one has the two inequalities

$$\int_{\Gamma}\frac{Q(y)^{(\alpha+1)}Q(y)^{(\beta+1)}}{Q(y+v)^{\frac{n}{2}+\alpha+\beta+2}}\phi(v)^{(p+1)'}dv\le C\phi(y)^{(p+1)'}$$

$$\int_{\Gamma}\frac{Q(y)^{(\alpha+1)}Q(y)^{(\beta+1)}}{Q(y+v)^{\frac{n}{2}+\alpha+\beta+2}}\phi(y)^{p+1}dv\le C\phi(v)^{p+1}$$

We use again Lemma (1.5) to conclude. As we have done in the introduction for the whole domain, we use the invariant measure to define the weighted spaces ,and note $L_v^p(\Gamma)$the space $L^p\left(\Gamma,Q^{v-\frac{n}{2}}dv\right)$.The unweighted case corresponds to the value $v=\frac{n}{2}$.

Using Lemma (1.6), we get necessary and sufficient conditions for the boundedness of the operator $S_{\alpha,\beta}$in the space $L_v^p(\Gamma)$ when we write that the kernel of the operator for this new measure, that is $\frac{Q(y)^{\alpha}Q(v)^{\beta-v+\frac{n}{2}}}{Q(y+v)^{\frac{n}{2}+\alpha+\beta}}$, has its p −th power integrable in the y variable , and its p' −th power integrable in the v variable .

Let us introduce the Laplace transform of a function g defined on Γ as

$$\mathcal{L}\{g(z)\}=\int_{\Gamma}e^{iZ.\xi}g(\xi)\frac{d\xi}{Q(\xi)^{\frac{n}{2}}}.$$

T he eigenvalues of $(a+1)^{-1}_{-2t}$ which are different from 1 are equal to $e^{\pm 2t}$. But

$$\lambda^{-2}e^{\pm 2t} \geq \frac{1}{x_n^2 - x_1^2}\frac{x_n - x_1}{x_n + x_1} = \frac{1}{(x_n + x_1)^2} \geq \frac{1}{2}.$$

The aim is to give L^p continuity properties for a family of operators on the cone itself which are closely related to the Bergman projection . Let us first consider integrability properties of products of powers of L and Q.

Lemma (1.5)[3]: Let $(\beta+1),(\gamma+1),(\mu+1)$ be real parameters. For $y \in \Gamma$, the integral

$$I_{(\beta+1),(\gamma+1),(\mu+1)}(y) = \int_\Gamma L(v)^{(\beta+1)} Q(y+v)^{(\gamma+1)} Q(v)^{(\mu+1)} dv .$$

is convergent if and only if the following conditions hold:

$\mu > -2$, $\gamma + \mu < \frac{-n-4}{2}$, $\beta + \mu > \frac{-n-4}{2}$, $\beta + \gamma + \mu < -n - 4$.

In this case, there is a positive constant $C_{(\beta+1),\gamma,\mu}$ such that for every $t \in \Gamma$:

$$I_{(\beta+1),(\gamma+1),(\mu+1)}(y) = C_{\beta,(\gamma+1),(\mu+1)} L(y)^{(\beta+1)} Q(y)^{\gamma+\mu+\frac{n}{2}+2}.$$

Proof : We give the proof for completeness since details are not given in [5]. Using the action of H, it is sufficient to assume that $y = \underline{e}$ .We consider the change of coordinates form $v \in \Gamma$ defined by

$$\tilde{v}_1 = v_n - v_1 , \qquad \tilde{v}_1 = v_n - v_1 - \frac{|\widetilde{v'}|^2}{v_n - v_1}$$

and $\widetilde{v'} = (\tilde{v}_3, \cdots, \tilde{v}_n) = (v_2, \cdots, v_n)$. It follows from an elementary computation that the determinant of the Jacobean is constant , that $Q(v) = \tilde{v}_1 \tilde{v}_2$ and

$$Q(v+\underline{e}) = (\tilde{v}_1 + 1)(\tilde{v}_2 + 1)\left(1 + \frac{|\widetilde{v'}|^2}{\tilde{v}_1(\tilde{v}_1+1)(\tilde{v}_2+1)}\right)$$

After a first integration in $\widetilde{v'}$, we are linked to consider the integral

$$\int_{v_1>0, v_2>0} v_1^{\beta+\mu+\frac{n-2}{2}} (v_1+1)^{\gamma+\frac{n-2}{2}} dv_1 d\, v_2.$$

The necessary and sufficient conditions follow at once . Let us remark that such coordinates exist in a more general setting , as proved in [26].

For positive $(\alpha+1),(\beta+1)$, let us now consider the integral operators $S_{\alpha,\beta}$ which are defined, on cone Γ by

$$S_{(\alpha+1),(\beta+1)} f(y) = \int_\Gamma \frac{Q(y)^{(\alpha+1)} Q(v)^{(\beta+1)}}{Q(y+v)^{\frac{n}{2}+\alpha+\beta+2}} f(v) dv . \tag{9}$$

at most N points y_j which are at $d-$ distance less than $\delta+1$ of $\underline{e}$, and such that the balls of $d-$radius $\frac{(\delta+1)}{2}$ centered at $\underline{e}$ the Euclidean distance and the invariant distance are comparable . So the existence of Nfollows elementary considerations on the volume. Let us remark that one can easily compute the volume , for the Lebesgue measure, of the balls B_j or B'_j. It is clearly a constant (depending only on($\delta+1$)) , when the volume is taken in terms of the invariant measure $Q(y)^{-\frac{n}{2}}dy$. Since Q is almost constant on each ball , we have the following property

$$\mathrm{Vol}(B_j) \approx \mathrm{Vol}(B'_j) \approx Q(y)^{\frac{n}{2}} \tag{8}$$

Such a decomposition of the cone Γ into almost disjoint balls can be seen as the analog of a Whitney decomposition .We will also have to compare the usual Euclidean balls and the balls related to the metric d, we will need the following lemma .

Lemma (1.4)[3]: There exists a constant $\delta>-1$ such that ,for $y\in\Gamma$ with $|y|<1/2$, the ball B of $d-$radius δ which is centered in y is contained in the set $\{x\in\Gamma: |x|<1\}$.

Proof : We give the proof in detail ,since it is not contained in [5]. It is sufficient to show that , for $x\in\Gamma$ such that $|x|<1$, one has the inequality $\langle\xi+1,\xi+1\rangle_x \geq \frac{|\xi+1|^2}{2}$. Indeed , let us take for granted this inequality , and conclude . If the ball B of $d-$radius ($\delta+1$) which is centered in y is not contained in the set $\{x\in\Gamma: |x|<1\}$,it means that there exists a geodesic ϕ which maps 0 into some point x with $|x|=1$, such that $|\phi(t)|<1$ for t in the interval (0,1) and such that

$$\int_0^1 \langle\phi'(t),\phi'(t)\rangle_{\phi(t)}^{\frac{1}{2}}dt \leq (\delta+1) \ .$$

Using the inequality between the infinitesimal metrics ,we see that the Euclidean length of ϕ is less than or equal to$\sqrt{2(\delta+1)}$ .So it is sufficient to take $\delta=1/4$ to get a contradiction . To show the inequality between the infinitesimal metrics ,we use the invariance of both sides when rotating in the $n-1$ first variables. It allows us to assume that $x=(x_1,0,\dots,0,x_n)$, with $x_1>0$. According to (7), we can write $x=\lambda(a+1)_t\underline{e}$, with $\lambda=(x_n^2-x_1^2)^{\frac{1}{2}}$ and ,Then

$$\langle\xi+1,\xi+1\rangle_x=\lambda^{-2}(\xi+1)\cdot(a+1)_{-2t}^{-\frac{1}{2}}(\xi+1).$$

where

$$\lambda = Q(y)^{\frac{1}{2}} \;;\quad t = ln\frac{Q(y)^{\frac{1}{2}}}{y_n - y_1} \;; v = -\frac{1}{y_n - y_1}\begin{pmatrix} y_2 \\ \vdots \\ y_{n-1} \end{pmatrix} \qquad (7)$$

So Γ may be identified with H. In this identified , the measure $Q(y)^{-\frac{n}{2}}dy$ on Γ which is invariant under the action of H , gives the Haar measure .

The quadratic form $Q(y)$ and the linear functional $L(y) = y_2 - y_1$ have invariance properties under the action of $g = \lambda n_t a_t \in H$:

$$Q(gy) = (detg)^{\frac{2}{n}} Q(y), \quad L(gy) = (detg)^{\frac{1}{n}} e^{-t} L(y) .$$

It is possible to define an H −invariant Riemannian metric d on Γ as follows : if $y = g\underline{e} \in \Gamma$ with $g \in H$, and ξ, η are tangent vectors at y, we set

$$\langle \xi, \eta \rangle_y = g^{-1}\xi \cdot g^{-1}\eta \cdot$$

By construction , the metric d is invariant under the action of H . It is easy to see that it is also invariant under the whole group $G(\Gamma)$ .This metric can be used to see that Γ is an almost disjoint of sets on which Q is basically constant .This follows from the two following lemmas.

Lemma(1.2)[3]: Given $\delta + 1 > 0$ and $k_0 > 0$ integer , there is $\gamma > 0$ such that if $B_1, \dots, B_K$ are balls in Γ of d −radius smaller than $\delta + 1$, with $k \le k_0$ then ,for $y, y' \in B_1 + \cdots + B_K$

$$\gamma^{-1} Q(y') < Q(y) < \gamma Q(y'),$$

We refer to [5] for its proof .

Lemma(1.3)[3]: There is a constant N such that , given $-1 < \delta < 0$, one can find a sequence of points $\{y\}$ in Γ with the property that , if we call B_j and B_j' the balls with center y_j and d −radius $\delta + 1$ and $\frac{(\delta+1)}{2}$ respectively , then

(i) the balls B_j' are pair wise disjoint;

(ii) the balls B_j cover Γ and are almost disjoint in the sense that each point .

Proof : We take $\{y_j\}$ a maximal subset of Γ (under inclusion) among those with the property that their elements are distant at least $\delta + 1$ one from the other . Clearly the balls B_j' are disjoint .If the B_j were not a covering of Γ ,this would contradict the maximality of the sequence . Let us show the finite overlapping property. We want to show that , for $y \in \Gamma$, there are at most N points y_j at d −distance less than $\delta + 1$ of y . By invariance of the distance , we may as well show that there are that the Bergman projection on L^{p+1}, which is not the case in general . Nevertheless their results are exact with additional assumptions , which we try to choose as weak as possible .Our proof for atomic decomposition is new , and can be generalized to all tube domains over homogeneous cones . The forward light cone Γ is a symmetric cone. Such cones have been studied by Gindikin in [11] . Nowadays , the book of Faraut and Koranyi [12] is a very good reference for their study . There , the cone Γ is called the Lorentz cone .

When we say that it is a symmetric cone , we mean that it satisfies the two properties :

(i) it is self-dual , which means that , for $x \in \mathbb{R}^n$, the scalar product x, y is strictly positive for every $y \in \Gamma$ if and only if x lies in Γ ;

(ii) it is homogeneous , which means that the subgroup $G(\Gamma)$ of $GL(\mathrm{n})$ leaving Γ invariant acts transitively on Γ . Here each element $g \in G(\Gamma)$whichbelongs to the connected component of the identity may be written as λg_0 ,with $\lambda > 0$ and g_0 in the Lorentz group special orthogonal group $(n-1{,}1)$, that is the group of $n \times n$ matrices of determinant 1 which preserve the quadratic form Q. It may be useful to know that Γ identifies to a subgroup of $\mathbb{R}_+ \times SO$ (special orthogonal group) $(n-1{,}1)$ (remember that $\mathbb{R}_+$ may also be considered as a group). More precisely, let us consider the two following subgroups of $SO(n-1{,}1)$ (they appear in the Iwasawa decomposition of the connected component of the identity $SO(n-1{,}1)$) :

(iii) the subgroup A consisting of the matrices

$$a_t = \begin{pmatrix} cosht & 0 & sinht \\ 0 & I_{n-2} & 0 \\ sinht & 0 & cosht \end{pmatrix}$$

with $t \in \mathbb{R}$;

(iv)the subgroup N consisting of the matrices

$$n_v = \begin{pmatrix} 1-\frac{|v|^2}{2} & -{}^t v & \frac{|v|^2}{2} \\ v & I_{n-2} & -v \\ -\frac{|v|^2}{2} & -{}^t v & 1+\frac{|v|^2}{2} \end{pmatrix}$$

with $v \in \mathbb{R}^{n-2}$.

Then , it can be shown that $H = R^+NA$ is a group , which acts simply transitively on Γ . If $\underline{e} = (0, \dots ,0,1)$ then each $y \in \Gamma$ can be written in a unique way as $y = \lambda n_v (a+1)_t \underline{e}$

Theorem (1.1).Property (ii) is elementary and does not depend on sophisticated counter-examples . One may conjecture that the range of p in (ii) is too large .We have been able to show that it is indeed the case when $n = 3$ and $v < 5/2$. For these values , the interval in (ii) may be replaced by the smaller interval

$1 + \frac{1}{4v-5} < p < 4(v-1)$ .We do not the proof here ,first because of its difficulty, and next because of the incompleteness of such an improvement . Let us remark that , nevertheless , when we consider the unwieghted case $v = n$, the result given in (iii) is asymptotically the best one for n tending to ∞ .On the other hand , as we shall see , the Szego projection corresponds to $v = \frac{n}{2}$. For this value (which is not in the range allowed in the statement of the theorem), the closure of the interval given in (iii) reduces to $\{2\}$. In this case , it has been proved by Fefferman unbounded for $p \neq 2$ (see [6]and [7]) .This gives an argument to conjecture that the interval given in (iii) coincides with the range of p for which the weighted Bergman projection is bounded . On the other hand , we did not find any possibility to adapt the counter-example in [6] for the Bergman projection .We generalize the results to all tube domains over symmetric cones in joint work with G. Garrigos in preparation [8] . It has already been partly done for (i) and (ii) in [9] . We will present here the results in a somewhat different way compared to [5] . First , the present section is organized so that all parts of the theorem are showed in the same way . In particular ,we adopt the same notations and kind of proof for (i) and (iii) . Secondly, an easy generalization of the proofs ,which we perform here, allows to get continuity properties when the Bergman projection P_v is replaced by the operator whose kernel is some complex power α of the Bergman kernel $B_v(z,w)$, with constant which increase exponentially with the imaginary part of $(\alpha + 1)$. Let us recall that consideration of an analytic family of operator is necessary when one wants to apply complex interpolation for different values of v. With this new point of view we do not need to refer to the D Alembert operator . In the last section, we give some applications for the Bergman spaces A_v^p: duality, sampling theorems , atomic decomposition , and restriction theorems . Let us recall that this kind of properties had been stated for the first time in the fundamental section of Coifman and Rochberg [10] . Unfortunately they had assumed phenomenon , compared to all cases for which the Bergman projection is known to satisfy L^{p+1} estimates so ,while the proof of (i) uses basically the same methods as in the upper half-plane ,that is Schur's Lemma which gives continuity properties for positive kernels , we must take advantage of the oscillations of the Bergman kernel to get the larger range of values of P given in (iii) ,We use two main ideas to do it in [5] . The first one is to show that L^p continuity is equivalent to some Hardy type inequality in the Bergman classes .The usual derivative ,which is for the classical Hardy inequality in the upper half-plane ,is replaced by the D'Alembert operator (or wave operator) given by

$$\square = Q((\partial+1)_z) = -(\partial+1)^2_{z_1} - \cdots - (\partial+1)^2_{z_{n-1}} + (\partial+1)^2_{z_n}. \quad (5)$$

It is a remarkable property that L^p continuity for $P_v$with $2 < p < \infty$ is equivalent to the fact that the D'Alembert operator satisfies the inequality

$$\|F\|_{A^p_v} \leq C \|\square^{\,m} F\|_{A^p_{v+mp}} \quad (6)$$

for m large enough. We are then linked to only consider holomorphic function F , which may be written as Laplace transforms of functions g defined on the cone Γ .Unfortunately , as in the classical case of the upper-half plane , there is no easy characterization of the fact that F belongs to A^p_v in terms of g . This last difficulty is the reason for a further generalization .If we write $z = x + iy$,we identify a function $f(z)$ on Ω with a function of two variables $x \in R^n$ and $y \in \Gamma$, and we note again $f(x+iy)$ for simplification .The measure $dV(z)$ is $dxdy$. Let us define $L^{p,q}_v = L^p(\Gamma, Q(y)^{v-n}dy, L^q(R^n, dx))$ as the space of functions $f(x+iy)$on Ω such that $\|f\|^p_{L^{p,q}} = \int_{L^{p,q}}\left(\int_{L^{p,q}} |f(x+iy)|^q\, dx\right)^{p/q} Q(y)^{v-n} dy$ is finite (with the obvious modification when $p = \infty$) . As before we call $A^{p,q}_v$,we get the previous weighted Bergman spaces . Let us remark that , in the classical case of the upper half-plane , $A^{p,q}_v$ identifies with a closed subspace of some Besov space at the boundary .We will consider continuity properties of P_v for the whole range of space $L^{p,q}_v$. The same equivalent formulation (6) is valid in this context. What is new for these space is the fact that we are able to give a complete answer for $q = 2$ .Indeed , the functions $F \in A^{p,q}_v$are completely characterized though their Laplace transforms , and we have ,for $0 < p < p_v$ for some critical index P_v , a Littlewood-Paley type characterization of the space $A^{p,q}_v$. Let us add some comments on

The condition on v corresponds to the integrability of the weight, When it is not satisfied , the Bergman space A_v^2 is reduced to $\{0\}$. Up to small changes , parts (i) and (ii) have been proved in [4] . Let us point out that results of [4] are given for the spherical cone which is defined in the same way , with Q and L replaced by $Q(y) = y_1 y_n - (y_2^2 + \cdots + y_{n-1}^2)$, $L(y) = y_1$. An elementary change of variable allows to pass from one to the other .The usual coordinates of the spherical cone have the advantage that direct computations are easy to perform , and most proofs in [4] are done by hand .We do not give here these elementary proofs but replace them by more sophisticated ones , using the geometry of the cone . They may be generalized more easily to all tube domains over symmetric cones .Moreover , only the unweighted case is considered in[4] , but everything generalizes to the weighted case .Let us specially mention the transfer principle , from which it follows that Bergman projections are bounded in the same time for both domains . There are two ingredients for its proof .One is the basic identity

$$B_{\widetilde{\Omega}}(z,w) = B_{\Omega,v}\big(\Phi(z),\Phi(w)\big) J_\Phi(z)\overline{J_\Phi(w)}, \quad (3)$$

where Φ is the holomorphism which maps $\widetilde{\Omega}$ onto Ω and J_Φ stands for the Jacobin of Φ fact that .The other one is the J_Φ is locally bounded far from its zeroes .Using the same identity for the weights ,one gets easily the weighted version of (3) :

$B_{\widetilde{\Omega},v}(z,w) = B_\Omega\big(\Phi(z),\Phi(w)\big) J_\Phi(z)^{\frac{v}{n}} \overline{J_\Phi(w)^{\frac{v}{n}}}$. From this , it is easy to that the transfer principle is also valid for the weighted Bergman projections . So, from now will only consider the unbounded domain Ω. The weigthed Bergman kernel of Ω can be written explicitly in terms of the complexified quadratic form Q as

$$B_v(z,w) = c_v(-Q)^{-v}(z - \overline{w}). \quad (4)$$

In particular , on the diagonal , if we use the notation $z = x + iy, B(z,z)$ is equal, up to a constant , to $Q(y)^{-n}$.So the weights $B_v(z,z)^{1-\frac{v}{n}}$ depend only of y , and are equal , up to constants , to $Q(y)^{v-n}$. We shall omit the constants from now on . Let us now give some comments on which is new , and has been obtained in a joint work with Marco Peloso and Fulvio Ricci [1].When considering simultaneously (i) and (iii) ,we see that there are values of p for which the Bergman projection P_v is bounded while the positive operator P_v^+ is not .This is a new

$$\Gamma = \{y = (y_1, \dots, y_{n-1}, y_n) \in R^n : Q(y) > 0, L(y) > 0\} \qquad (1)$$

Here Q is the quadratic from $Q(y) = -y_1^2 + y \dots - y_{n-1}^2 + y_{n-1}^2 + y_n^2$, and Q is the linear form $L(y) = -y_1 + y_n$. We could as well have given $y_n > 0$ as a second condition .We introduce the linear form L right now since it will play a role later .The fact that the cone is defined by two inequalities is related to its rank ,which is equal to 2. The domain Ω is an affine-homogeneous symmetric tube domain known as the Lie the ball $\widetilde{\Omega}$ of C^n .This last one is defined by

$$\widetilde{\Omega} = \left\{z \in C^n : \left|\sum_j^n z_j^2\right| < 1, 1 - 2|z|^2 + \left|\sum_j^n z_j^2\right|^2 > 0\right\} \qquad (2)$$

In Elie Cartans classification of bounded symmetric domains [1], the lie balls are the representative of class IV (according to Huas numbering [2]).

It turns out that, even if we are only interested in L^{p+1} continuity of the Bergman projection P (when D is Ω or $\widetilde{\Omega}$) , it is relevant to consider weighted projections as well .We define L_v^p as the Lebesgue space for the weighted measure $B(z,z)^{1-\frac{v}{n}}\, dV\,(z)$. The unweighted case corresponds to the value $v = n$ .Next we define the weighted Bergman spaces A_v^p , and the orthogonal projection P_v of L_v^p onto A_v^p .Moreover ,we adopt the notation B_v for the corresponding weighted Bergman kernel ,and we call P_v^+ the positive integral operator defined by

$$P_v^+ f(z) = \int_D |B_v(z,w)|\, f(w) B(w,w)^{1-\frac{v}{n}} dv(w).$$

We can now state the L^p continuity results for the Bergman projections of the two domains that we consider .

Keyword : , Bergman projections , Laplace transforms , Cone , Operator , Bergman space , D'Alembert operator , Lebesgue measure , Decomposition

Theorem (1.1)[3]: For the domains Ω and $\widetilde{\Omega}$,and for $v > n - 1$, one has the following properties .

(i) The operator P_v^+ is bounded on L_v^p if and only if $\frac{2(v-1)}{2v-n} < p < \frac{2(v-1)}{n-2}$.

(ii) If P_v is bounded on L_v^p , then $1 + \frac{n-2}{2v} < p < 1 + \frac{2v}{n-2}$.

(iii) P_v is bounded on L_v^p for $1 + \frac{n-2}{2(v-1)} < p < 1 + \frac{2(v-1)}{n-2}$.

Abstract :
In this paper , shall concentrate on new L^{p+1} continuity properties for an analytic family of operations include the Bergman projection of the tube domains over the of light cone . These last results extend to the bounded realization of tube domains under consideration .

المستخلص :
هذه الدراسة ستركز علي استمرارية جديدة لخصائص فضاء ليبيق الممتد لأجل عائلة تحليلية من عمليات التي تشمل خصائص بار قمن من مجالات الأنبوبة الفوقية لمخروط الضوء . هذه النتائج الأخيرة لتمديد المحدود تحقيق اعتبار تحت مجالات الأنبوبة .

Introduction :

For D a domain in C^n and $p+1 \in (0,\infty]$,we denote by L^{p+1} the Lebesgue spaces $L^{p+1}(D,dV)$ (where dV stands for the Lebesgue measure on C^n) , and by A^{p+1} the Bergman space, consisting of holomorphic functions on D which belong to L^{p+1} .It is well known, and elementary ,that A^{p+1} is a closed subspace of L^{p+1} .The Bergman projection P is the orthogonal projection of L^2 on A^2 . It is given by

$$Pf(z) = \int_D B(z,w)f(w)dV(w)$$

where $B(z,w)$ is the Bergman kernel of D .We write $B_D(z,w)$ it is necessary for comprehension ,but do not specify the domain when there is no ambiguity . Continuity properties of the Bergman projection have been widely studied when D is a pseudo-convex domain with smooth boundary . Here we deal with other kinds of domains ,which are simplest ones among the Siegel domains of type 11 of rank larger than one , and consider L^{p+1} −continuity properties . It turns out that the range of $p+1$ for which $P+1$ extends to a bounded operator on L^{p+1} is unknown ,and we give partial results in this direction. We will focus our attention on two specific domains : the first one is the complex tube domain $\Omega = R^n + i\Gamma \subset C^n$, $n \geq 3$, where Γ is the forward light cone given by

GENERALIZATION OF SUPERSTRUCTURE DOMAINS FOR LIGHT CONES IN LOBAGE EXTENDED SPACE

Dr. Sami Ali Ahmed AL-hag*

* Assistant Proffesor ,Department of physics & Mathematics, Faculty of Education, University of AL–Butana